MODERN ENGINEERING
&
MANAGEMENT STUDIES

SUBJECT NAME – OPTIMIZATION IN ENG.
SUBJECT CODE – ROE6A001
LECTURE NOTES
B.TECH 3rd YEAR – SEM- VI (2024-2025)

DEPARTMENT OF BASIC SCIENCE
AND
HUMANITIE

MODULE- II

In operations management, both transportation and assignment problems are about optimizing resource allocation, but they differ in their scope and constraints. Transportation problems focus on distributing goods or resources from multiple sources to multiple destinations to minimize total cost, while assignment problems deal with assigning tasks or jobs one-to-one to minimize overall costs.

Transportation Problem:

- **Objective:**

  Minimize the total cost of transporting goods from multiple sources (origins) to multiple destinations (targets) while satisfying supply and demand constraints.

- **Key Features:**
- Involves multiple sources and destinations.
- The goal is to determine the optimal quantity of goods to ship from each source to each destination.
- Supply and demand constraints must be met at each source and destination.
- Transportation costs are known for each source-destination route.
- **Types:**
- **Balanced:** Total supply equals total demand.
- **Unbalanced:** Total supply does not equal total demand, requiring the introduction of dummy sources or destinations.
- **Methods:**
- **North-West Corner Method:** A method for finding an initial feasible solution.
- **Least Cost Method:** Another method for finding an initial feasible solution.
- **Vogel's Approximation Method:** A method for finding an initial feasible solution.
- **MODI (Modified Distribution) Method:** Used to find the optimal solution.
- **Applications:**
- Shipping goods from warehouses to retailers.
- Distributing resources in a logistics network.
- Optimizing transportation routes.

Assignment Problem:

- **Objective:**

  Minimize the total cost of assigning tasks or jobs to individuals or resources, one-to-one.

- **Key Features:**
- Involves a one-to-one matching between sources and destinations.
- Each source can be assigned to only one destination.
- Each destination can be assigned to only one source.

- **Methods:**
- **Hungarian Method:** A widely used method for solving assignment problems.
- **Applications:**
- Assigning employees to projects.
- Matching patients to doctors.
- Assigning tasks to machines.

Relationship between Transportation and Assignment Problems:

- The assignment problem is a special case of the transportation problem, where the number of sources and destinations are equal, and each source can only be assigned to one destination.
- Both problems are linear programming problems, meaning the relationship between variables and constraints is linear.

1)) NWCM:

This method starts at the north west (upper left) corner cell of the tableau (variable x_{11}).

Step 1: Allocate as much as possible to the selected cell, and adjust the associated amounts of capacity (supply) and requirement (demand) by subtracting the allocated amount.

Step 2: Cross out the row (column) with zero supply or demand to indicate that no further assignments can be made in that row (column). If both the row and column becomes zero in the same time, cross out one of them only, and leave a zero supply or demand in the uncrossed out row (column).

Step 3: If exactly one row (column) is left uncrossed out, then stop. Otherwise, move to the cell to the right if a column has just been crossed or the one below if a row has been crossed out. Go to step 1.

Refer to above example:

	Retail Agency					
Factories	1	2	3	4	5	Capacity
1	1	9	13	36	51	50
2	24	12	16	20	1	100
3	14	33	1	23	26	150
Requirement	100	60	50	50	40	300

Retail Agency / **Factories**	1	2	3	4	5	**Capacity**

1	1 **50**	9	13	36	51	~~50~~
2	24 **50**	12 **50**	16	20	1	~~100~~ ~~50~~
3	14	33 **10**	1 **50**	23 **50**	26 **40**	~~150~~ ~~140~~ ~~90~~ ~~40~~
Requirement	~~100~~ ~~50~~	~~60~~ ~~10~~	~~50~~	~~50~~	~~40~~	

The arrows show the order in which the allocated amounts are generated. The starting basic solution is given as

$x_{11} = 50$, $x_{21} = 50$, $x_{22} = 50$, $x_{32} = 10$, $x_{33} = 50$, $x_{34} = 50$, $x_{35} = 40$.

The corresponding transportation cost is:

$Z= 50 * 1 + 50 * 24 + 50 * 12 + 10 * 33 + 50 * 1 + 50 * 23 + 40 * 26 = 4420$

It is clear that as soon as a value of x_{ij} is determined, a row (column) is eliminated from further consideration. The last value of x_{ij} eliminates both a row and column. Hence a feasible solution computed by North West Corner Method can have at most $\mathbf{m + n - 1}$ positive x_{ij} if the transportation problem has **m** sources and **n** destinations.

2)) LCM

Least cost method is also known as matrix minimum method in the sense we look for the row and the column corresponding to which C_{ij} is minimum. This method finds a better initial basic feasible solution by concentrating on the cheapest routes. Instead of starting the allocation with the northwest cell as in the North West Corner Method, we start by allocating as much as possible to the cell with the smallest unit cost. If there are two or more minimum costs then we should select the row and the column corresponding to the lower numbered row. If they appear in the same row we should select the lower numbered column. We then cross out the satisfied row or column, and adjust the amounts of capacity and requirement accordingly. If both a row and a column is satisfied simultaneously, only one is crossed out. Next, we look for the uncrossed-out cell with the smallest unit cost and repeat the process until we are left at the end with exactly one uncrossed-out row or column.

Retail Agency / Factories	1	2	3	4	5	**Capacity**
1	1 **50**	9 **X**	13 **X**	36 **X**	51 **X**	**50**
2	24 **X**	12 **60**	16 **X**	20 **X**	1 **40**	**100**
3	14 **50**	33 **X**	1 **50**	23 **50**	26 **X**	**150**
Requirement	**100**	**60**	**50**	**50**	**40**	

We observe that C_{11}=1 is the minimum unit cost in the table. Hence X_{11}=50 and the first row is crossed out since the row has no more capacity. Then the minimum unit cost in the uncrossed-out row and column is C_{25}=1, hence X_{25}=40 and the fifth column is crossed out. Next C_{33}=1is the minimum unit cost, hence X_{33}=50 and the third column is crossed out. Next C_{22}=12 is the minimum unit cost, hence X_{22}=60 and the second column is crossed out. Next we look for the uncrossed-out row and column now C_{31}=14 is the minimum unit cost, hence X_{31}=50 and crossed out the first column since it was satisfied. Finally C_{34}=23 is the minimum unit cost, hence X_{34}=50 and the fourth column is crossed out.

So that the basic feasible solution developed by the Least Cost Method has transportation cost is

Z= 1 * 50 + 12 * 60 + 1 * 40 + 14 * 50 + 1 * 50 + 23 * 50 = 2710

3)) Vogel Approximation Method (VAM)

Step 1: For each row and column find the difference between the two lowest unit shipping costs.

Step 2: Assign as many units as possible to the lowest-cost square in the row and column selected.

Step 3: Eliminate the column or row that has been satisfied.

Example: Consider the following transportation problem

Origin	Destination				
	1	2	3	4	a_i
1	20	22	17	4	120
2	24	37	9	7	70
3	32	37	20	15	50
b_j	60	40	30	110	240

Solution:

1. Compute the penalty for various rows and columns.

2. Look for the highest penalty in the row or column, the highest penalty occurs in the second column and the minimum unit cost i.e. c_{ij} in this column is c_{12}=22. Hence assign 40 to this cell i.e. x_{12}= 40 and cross out the second column (since second column was satisfied).

Origin	Destination					
	1	2	3	4	a_i	**column Penalty**

1	20 **X**	22 40	17 **X**	4 80	120 **13**
2	24	37 **X**	9 30	7	70 **2**
3	32	37 **X**	20 **X**	15	50 **5**
b_j Row Penalty	60 **4**	40 **15**	30 **8**	110 **3**	240

Origin	Destination				a_i **column Penalty**
	1	2	3	4	
1	20 **X**	22 40	17 **X**	4 80	120 **13**
2	24 10	37 **X**	9 30	7 30	70 **2** **17**
3	32 50	37 **X**	20 **X**	15 **X**	50 **5** **17**
b_j Row Penalty	60 **4** **8**	40 **15**	30 **8**	110 **3** **8**	240

The transportation cost corresponding to this choice of basic variables is:

Z = 22 * 40 + 4 * 80 + 30 * 9 + 30 * 7 + 10 * 24 + 50 * 32 = 3520.

3. Transportation Problem (Vogel's Approximation Method)

The <u>North-West Corner</u> method and the <u>Least Cost Cell</u> method has been discussed in the previous articles. In this article, the **Vogel's Approximation** method will be discussed.

Destination

Source	D1	D2	D3	D4	Supply
O1	3	1	7	4	300
O2	2	6	5	9	400
O3	8	3	3	2	500
Demand:	250	350	400	200	1200

Solution:

- For each row find the least value and then the second least value and take the absolute difference of these two least values and write it in the corresponding row difference as shown in the image below. In row **O1**, **1** is the least value and **3** is the second least value and their absolute difference is **2**. Similarly, for row **O2** and **O3**, the absolute differences are **3** and **1** respectively.
- For each column find the least value and then the second least value and take the absolute difference of these two least values then write it in the corresponding column difference as shown in the figure. In column **D1**, **2** is the least value and **3** is the second least value and their absolute difference is **1**. Similarly, for column **D2**, **D3** and **D3**, the absolute differences are **2**, **2** and **2** respectively.

Destination

Source	D1	D2	D3	D4	Supply	Row Difference
O1	3	1	7	4	300	2
O2	2	6	5	9	400	3
O3	8	3	3	2	500	1
Demand:	250	350	400	200	1200	
Column Difference:	1	2	2	2		

- These value of row difference and column difference are also called as penalty. Now select the maximum penalty. The maximum penalty is **3** i.e. row **O2**. Now find the cell with the least cost in row **O2** and allocate the minimum among the supply of the respective row and the demand of the respective column. Demand is smaller than the supply so allocate the column's demand i.e. **250** to the cell. Then cancel the column **D1**.

Destination

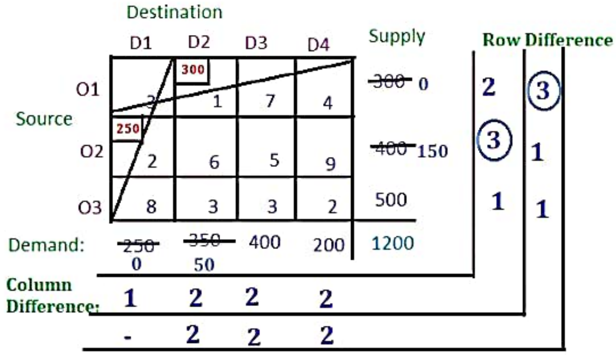
Source	D1	D2	D3	D4	Supply	Row Difference
O1	3	1	7	4	300	2
O2	250 2	6	5	9	~~400~~ 150	3
O3	8	3	3	2	500	1
Demand:	~~250~~ 0	350	400	200	1200	
Column Difference:	1	2	2	2		

- From the remaining cells, find out the row difference and column difference.

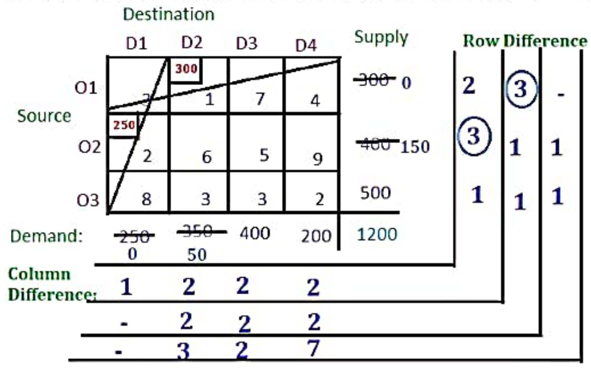
Destination

Source	D1	D2	D3	D4	Supply	Row Difference	
O1	3	1	7	4	300	2	3
O2	250 2	6	5	9	~~400~~ 150	3	1
O3	8	3	3	2	500	1	1
Demand:	~~250~~ 0	350	400	200	1200		
Column Difference:	1	2	2	2			
	-	2	2	2			

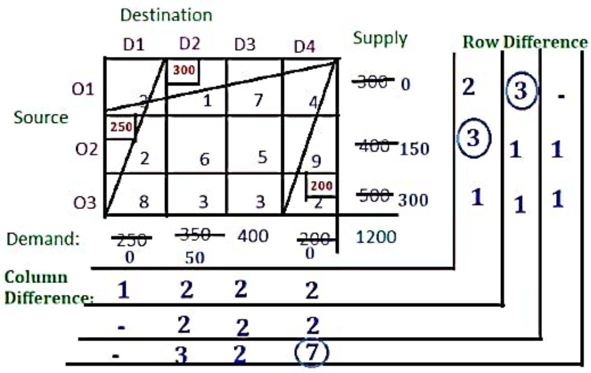
- Again select the maximum penalty which is **3** corresponding to row **O1**. The least-cost cell in row **O1** is **(O1, D2)** with cost **1**. Allocate the minimum among supply and demand from the respective row and column to the cell. Cancel the row or column with zero value.



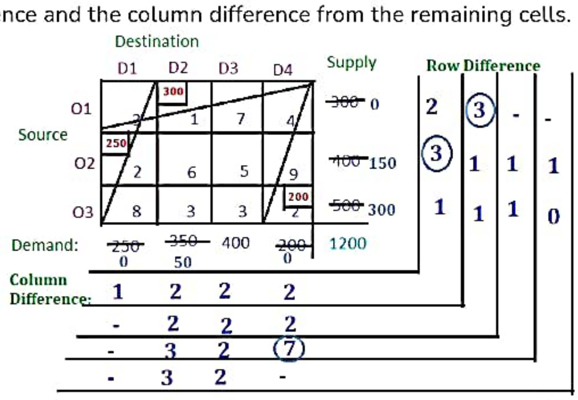
- Now find the row difference and column difference from the remaining cells.



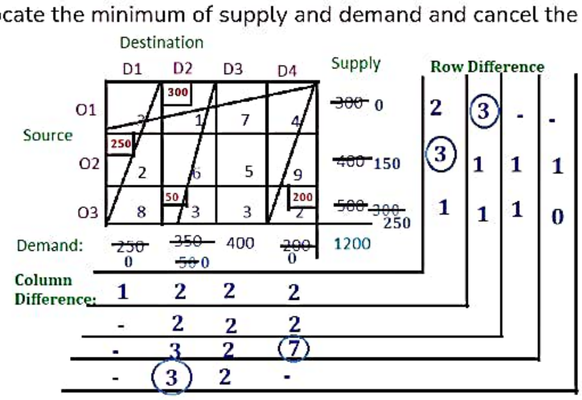
- Now select the maximum penalty which is **7** corresponding to column **D4**. The least cost cell in column **D4** is **(O3, D4)** with cost **2**. The demand is smaller than the supply for cell **(O3, D4)**. Allocate **200** to the cell and cancel the column.



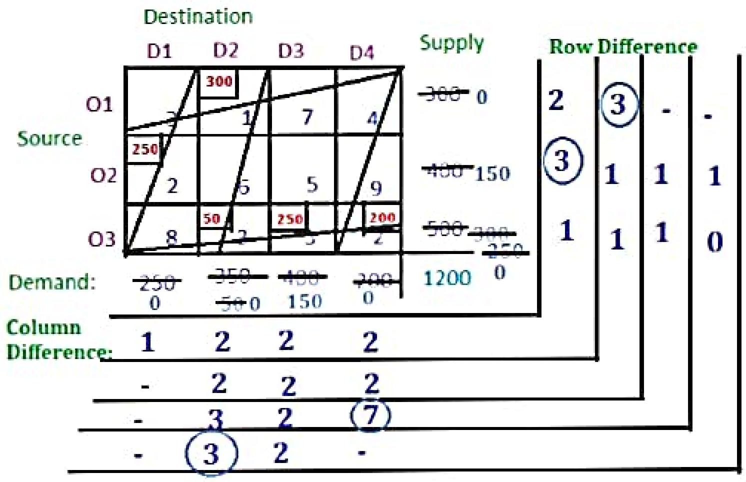
- Find the row difference and the column difference from the remaining cells.



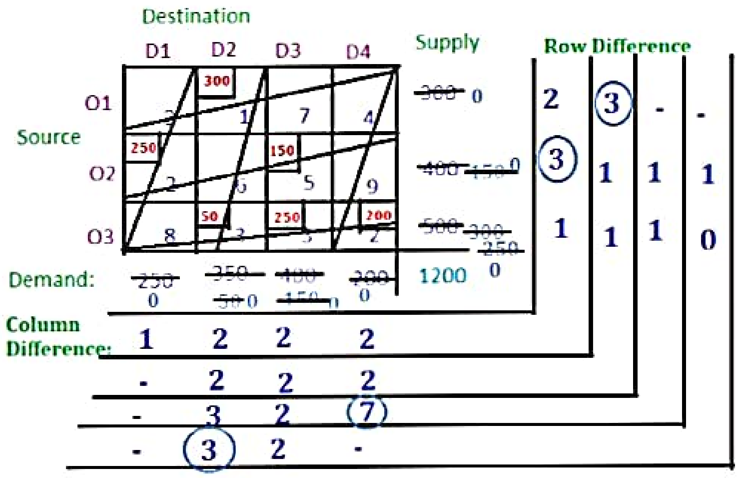
- Now the maximum penalty is **3** corresponding to the column **D2**. The cell with the least value in **D2** is **(O3, D2)**. Allocate the minimum of supply and demand and cancel the column.



- Now there is only one column so select the cell with the least cost and allocate the value.



- Now there is only one cell so allocate the remaining demand or supply to the cell



- No balance remains. So multiply the allocated value of the cells with their corresponding cell cost and add all to get the final cost i.e. **(300 * 1) + (250 * 2) + (50 * 3) + (250 * 3) + (200 * 2) + (150 * 5) = 2850**

4. Modified Distribution Method (MODI) or (u - v) Method:

The modified distribution method, also known as MODI method or (u - v) method provides a minimum cost solution to the transportation problem. In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Note: MODI method is an improvement over stepping stone method.

Steps in Modified Distribution Method (MODI)

1. Determine an initial basic feasible solution using any one of the three methods given below:
    - i. North West Corner Rule
    - ii. Matrix Minimum Method
    - iii. Vogel Approximation Method
2. Determine the values of dual variables, u_i and v_j, using $u_i + v_j = c_{ij}$
3. Compute the opportunity cost using $c_{ij} - (u_i + v_j)$.
4. Check the sign of each opportunity cost. If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. ***On the other hand, if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.***
5. Select the unoccupied cell with the most negative opportunity cost as the cell to be included in the next solution.
6. Draw a closed path or loop for the unoccupied cell selected in the previous step. ***Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.***
7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.
8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, add this quantity to all the cells on the corner points of the closed path marked with plus signs, and subtract it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.
9. Repeat the whole procedure until an optimal solution is obtained.

MODI Method Examples: Transportation Problem

In the previous section, we provided the steps in MODI method (modified distribution method) to solve a transportation problem. In this section, we provide an example. Let's solve the following example:

Consider the transportation problem presented in the following table.

		Distribution centre				
		D1	D2	D3	D4	Supply
Plant	P1	19	30	50	12	7
	P2	70	30	40	60	10
	P3	40	10	60	20	18
Requirement		5	8	7	15	

Determine the optimal solution of the above problem.

Solution:

An initial basic feasible solution is obtained by Matrix Minimum Method and is shown in table 1.

Table 1

		Distribution centre				
		D1	D2	D3	D4	Supply
Plant	P1	19	30	50	12 (7)	7
	P2	70 (3)	30	40 (7)	60	10
	P3	40 (2)	10 (8)	60	20 (8)	18
Requirement		5	8	7	15	

Initial basic feasible solution

12 X 7 + 70 X 3 + 40 X 7 + 40 X 2 + 10 X 8 + 20 X 8 = Rs. 894.

Calculating ui and vj using ui + vj = cij
Substituting u1 = 0, we get
$u_1 + v_4 = c_{14} \Rightarrow 0 + v_4 = 12$ or $\mathbf{v_4 = 12}$
$u_3 + v_4 = c_{34} \Rightarrow u_3 + 12 = 20$ or $\mathbf{u3 = 8}$
$u_3 + v_2 = c_{32} \Rightarrow 8 + v_2 = 10$ or $\mathbf{v_2 = 2}$
$u_3 + v_1 = c_{31} \Rightarrow 8 + v_1 = 40$ or $\mathbf{v_1 = 32}$
$u_2 + v_1 = c_{21} \Rightarrow u_2 + 32 = 70$ or $\mathbf{u_2 = 38}$
$u_2 + v_3 = c_{23} \Rightarrow 38 + v_3 = 40$ or $\mathbf{v_3 = 2}$
Table 2

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19	30	50	12 7	7	0
	P2	70 3	30	40 7	60	10	38
	P3	40 2	10 8	60	20 8	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Calculating opportunity cost using $c_{ij} - (u_i + v_j)$

Unoccupied cells	Opportunity cost
(P_1, D_1)	$c_{11} - (u_1 + v_1) = 19 - (0 + 32) = -13$
(P_1, D_2)	$c_{12} - (u_1 + v_2) = 30 - (0 + 2) = 28$
(P_1, D_3)	$c_{13} - (u_1 + v_3) = 50 - (0 + 2) = 48$
(P_2, D_2)	$c_{22} - (u_2 + v_2) = 30 - (38 + 2) = -10$
(P_2, D_4)	$c_{14} - (u_2 + v_4) = 60 - (38 + 12) = 10$
(P_3, D_3)	$c_{33} - (u_3 + v_3) = 60 - (8 + 2) = 50$

Distribution centre							
		D1	D2	D3	D4	Supply	u_i
Plant	P1	-13 19	28 30	48 50	12 7	7	0
	P2	70 3	-10 30	40 7	10 60	10	38
	P3	40 2	10 8	50 60	20 8	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Now choose the most negative value from opportunity cost (i.e., –13) and draw a closed path from P1D1. The following table shows the closed path.

		Distribution centre					
		D1	D2	D3	D4	Supply	u_i
Plant	P1	[-13] 19 +	[28] 30	[48] 50	- 12 (7)	7	0
	P2	70 (3)	[-10] 30	40 (7)	[10] 60	10	38
	P3	40 (2) -	10 (8)	[50] 60	+ 20 (8)	18	8
Requirement		5	8	7	15		
v_j		32	2	2	12		

Choose the smallest value with a negative position on the closed path(i.e., 2), it indicates the number of units that can be shipped to the entering cell. Now add this quantity to all the cells on the corner points of the closed path marked with plus signs and subtract it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

Now again calculate the values for u_i & v_j and opportunity cost. The resulting matrix is shown below.

		Distribution centre					
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19 (2)	[28] 30	[61] 50	12 (5)	7	0
	P2	70 (3)	[-23] 30	40 (7)	[-3] 60	10	51
	P3	[13] 40	10 (8)	[63] 60	20 (10)	18	8
Requirement		5	8	7	15		
v_j		19	2	-11	12		

Choose the most negative value from opportunity cost (i.e., –23). Now draw a closed path from P2D2 .

		Distribution centre					
		D1	D2	D3	D4	Supply	u_i
Plant	P1	+ 19 (2)	[28] 30	[61] 50	- 12 (5)	7	0
	P2	- 70 (3)	[-23] 30 +	40 (7)	[-3] 60	10	51
	P3	[13] 40	- 10 (8)	[63] 60	+ 20 (10)	18	8
Requirement		5	8	7	15		
v_j		19	2	–11	12		

Now again calculate the values for u_i & v_j and opportunity cost

		Distribution centre					
		D1	D2	D3	D4	Supply	u_i
Plant	P1	19 (5)	[28] 30	[38] 50	12 (2)	7	0
	P2	[23] 70	30 (3)	40 (7)	[20] 60	10	28
	P3	[13] 40	10 (5)	[40] 60	20 (13)	18	8
Requirement		5	8	7	15		
v_j		19	2	12	12		

Since all the current opportunity costs are non–negative, this is the optimal solution. The minimum transportation cost is: 19 X 5 + 12 X 2 + 30 X 3 + 40 X 7 + 10 X 5 + 20 X 13 = Rs. 799

Branch and Bound method

Consider a generic optimization problem

$$\min\{ c(\underline{x}) : \underline{x} \in X \}$$

Idea: Reduce the solution of a difficult problem to that of a sequence of simpler subproblems by (recursive) partition of the feasible region X.

Applicable to discrete and continuous optimization problems.

Two main components: branching and bounding.

$$z = \min\{ c(\underline{x}) : \underline{x} \in X \}$$

Branching:

Partition X into k subsets

$$X = X_1 \cup \ldots \cup X_k \text{ (with } X_i \cap X_j = \varnothing \text{ for each pair i} \neq \text{j)}$$

and let

$$z_i = \min\{ c(\underline{x}) : \underline{x} \in X_i \} \quad \text{for } i = 1, \ldots, k.$$

Clearly $z = \min\{ c(\underline{x}) : \underline{x} \in X \} = \min\{z_1, \ldots, z_k\}$

Bounding technique:

For each subproblem $z_i = \min\{ c(\underline{x}) : \underline{x} \in X_i \}$

i) determine an optimal solution of $\min\{ c(\underline{x}) : \underline{x} \in X_i \}$ (explicit), or

ii) prove that $X_i = \emptyset$ (explicit), or

iii) prove that $z_i \geq z'$ = objective function value of the best feasible solution found so far (implicit)

If the subproblem is not "solved" we generate new subproblems by further partition.

5.1.1 Branch and Bound for ILP

Given an ILP $\min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0} \text{ integer} \}$

Branching:

Partition the feasible region X into subregions (subdivision in exhaustive and exclusive subregions).

Achieved by solving the linear relaxation of the ILP

$$\min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0} \}$$

denote by $\overline{\underline{x}}$ an optimal solution and $z_{LP} = \underline{c}^T \overline{\underline{x}}$ the optimal value.

If $\underline{\bar{x}}$ integer, $\underline{\bar{x}}$ is also optimal for ILP, otherwise

$\exists\, \bar{x}_h$ fractional and we consider the two subproblems:

ILP$_1$: $\min\{\, \underline{c}^T\underline{x} : A\underline{x} = \underline{b},\ x_h \leq \lfloor \bar{x}_h \rfloor,\ \underline{x} \geq \underline{0} \text{ integer} \,\}$

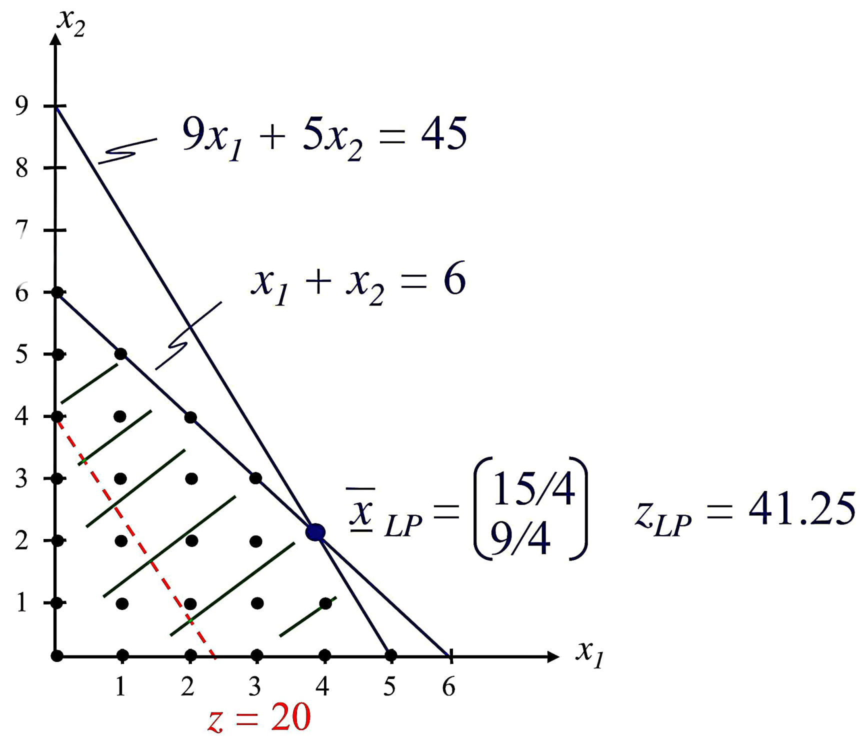
ILP$_2$: $\min\{\, \underline{c}^T\underline{x} : A\underline{x} = \underline{b},\ x_h \geq \lfloor \bar{x}_h \rfloor + 1,\ \underline{x} \geq \underline{0} \text{ integer} \,\}$

Bounding:

Determine a lower "bound" (if minimization ILP) on the optimal value z_i of a subproblem of ILP by solving its linear relaxation.

Example:

$$\text{(ILP)}\quad \begin{aligned} \max \quad & z = 8x_1 + 5x_2 \\ & x_1 + x_2 \le 6 \\ & 9x_1 + 5x_2 \le 45 \\ & x_1, x_2 \ge 0 \text{ integer} \end{aligned}$$



$$z_{LP} \ge z_{ILP}$$

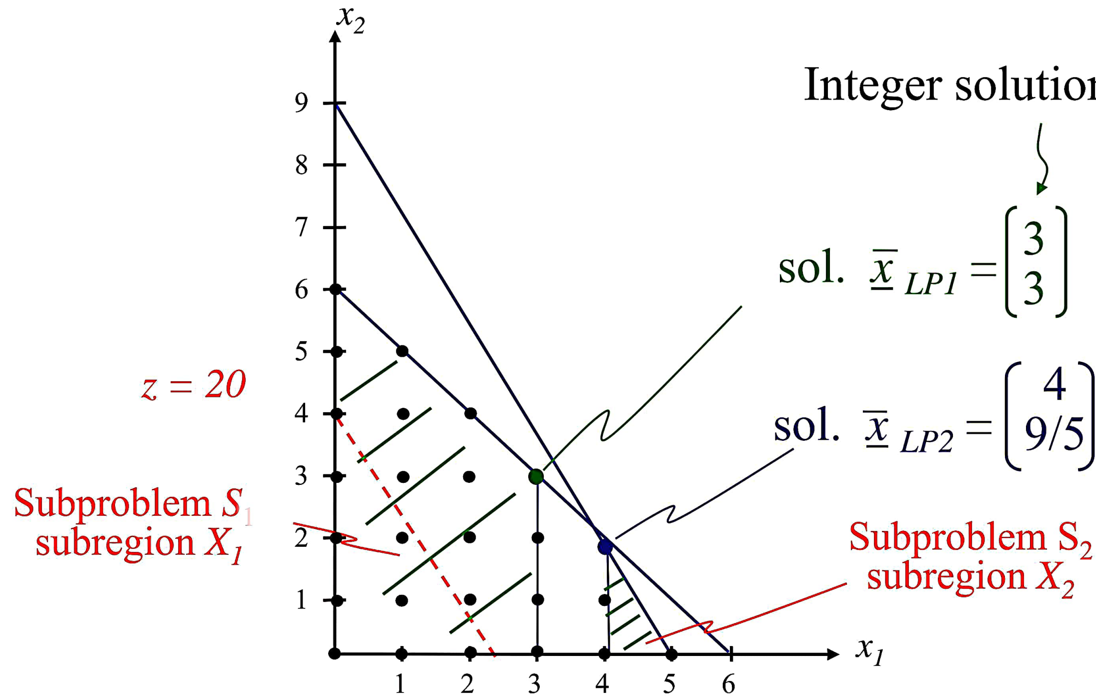
Since $\overline{x_1}$ and $\overline{x_2}$ are fractional, select one of them for branching.

for instance x_1

The feasible region X is partitioned into X_1 and X_2 by imposing:

$$x_1 \leq \lfloor \overline{x}_1 \rfloor = 3 \quad \text{or} \quad x_1 \geq \lfloor \overline{x}_1 \rfloor + 1 = 4$$

<u>exhaustive</u> and <u>exclusive</u> constraints



Integer solution ! $\Rightarrow z_{ILP1} = z_{LP1}$

sol. $\underline{\overline{x}}_{LP1} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ with $z_{LP1} = 39$

sol. $\underline{\overline{x}}_{LP2} = \begin{bmatrix} 4 \\ 9/5 \end{bmatrix}$ with $z_{LP2} = 41$

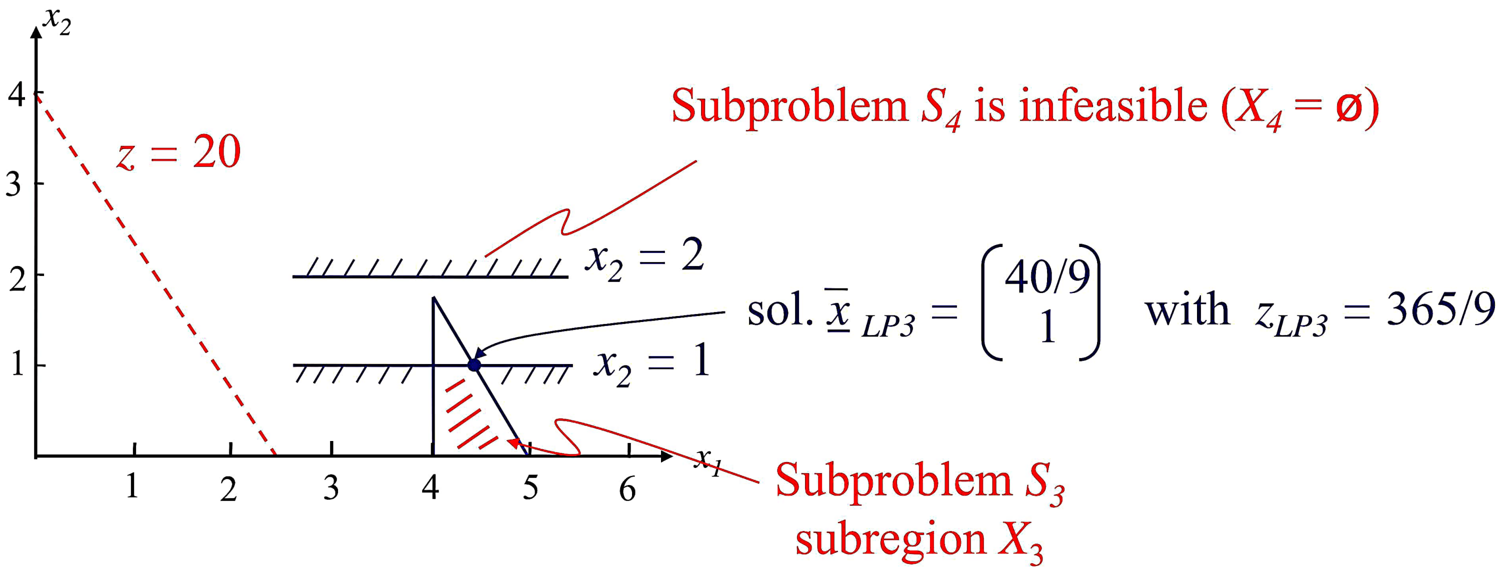
After considering X_1, best feasible (integer) solution found so far:

$$\underline{x}' = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \text{ with } z' = 39.$$

Since $z_{LP2} = 41 > 39$, X_2 may contain a better feasible solution of ILP.

$\Rightarrow$ Partition X_2 into X_3 and X_4 by imposing:

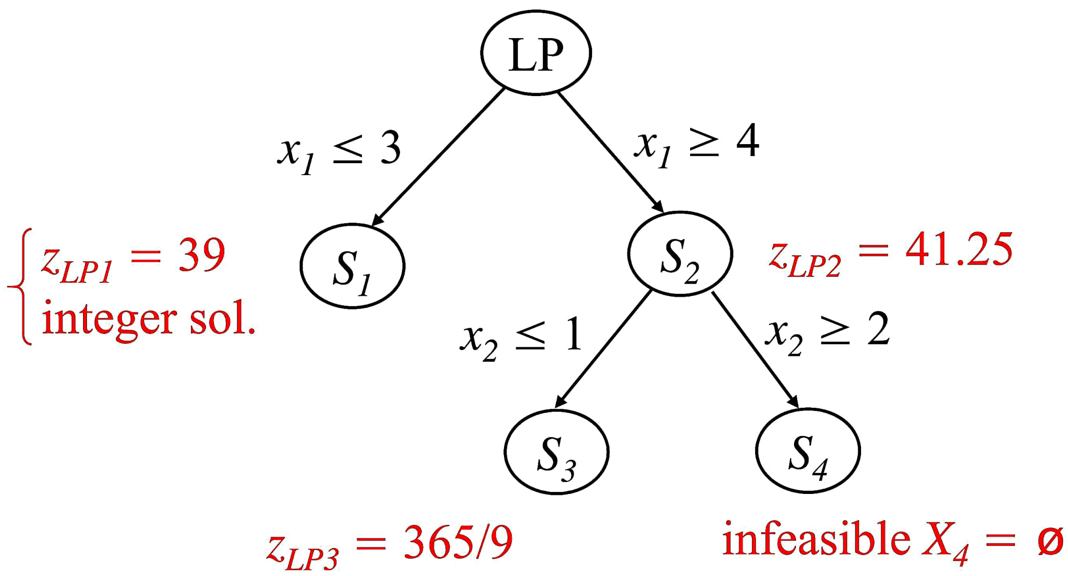
$$x_2 \leq \lfloor \bar{x}_2 \rfloor = 1 \quad \text{or} \quad x_2 \geq \lceil \bar{x}_2 \rceil + 1 = 2$$



x_2
4
3
2
1
$z = 20$
Subproblem S_4 is infeasible ($X_4 = ø$)
$x_2 = 2$
sol. $\underline{\bar{x}}_{LP3} = \begin{pmatrix} 40/9 \\ 1 \end{pmatrix}$ with $z_{LP3} = 365/9$
$x_2 = 1$
1
2
3
4
5
6
x_1
Subproblem S_3
subregion X_3

Branching tree:

Best feasible solution found so far

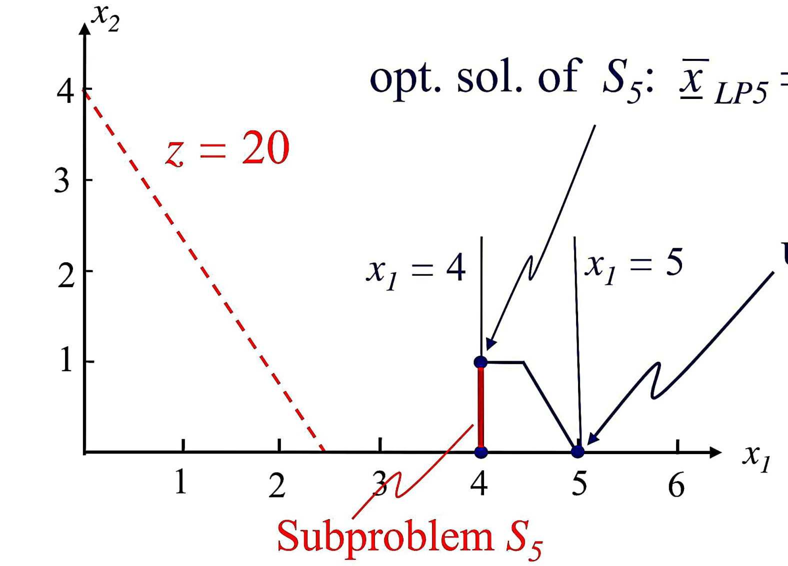


Since $z_{LP3} = 365/9 > 39$, X_3 may contain a better feasible solution of ILP.

$\Rightarrow$ <u>Partition</u> X_3 into X_5 and X_6 by imposing:

$$\underline{\bar{x}}_{LP3} = \begin{pmatrix} 40/9 \\ 1 \end{pmatrix} \qquad x_1 \leq \lfloor \bar{x}_1 \rfloor = 4 \quad \text{or} \quad x_1 \geq \lfloor \bar{x}_1 \rfloor + 1 = 5$$

opt. sol. of S_5: $\overline{\underline{x}}_{LP5} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ integer with $z_{LP5} = 37$



unique feasible solution of S_6

$$\begin{cases} \overline{\underline{x}}_{LP6} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad \text{integer} \\ z_{LP6} = \underline{40} \end{cases}$$

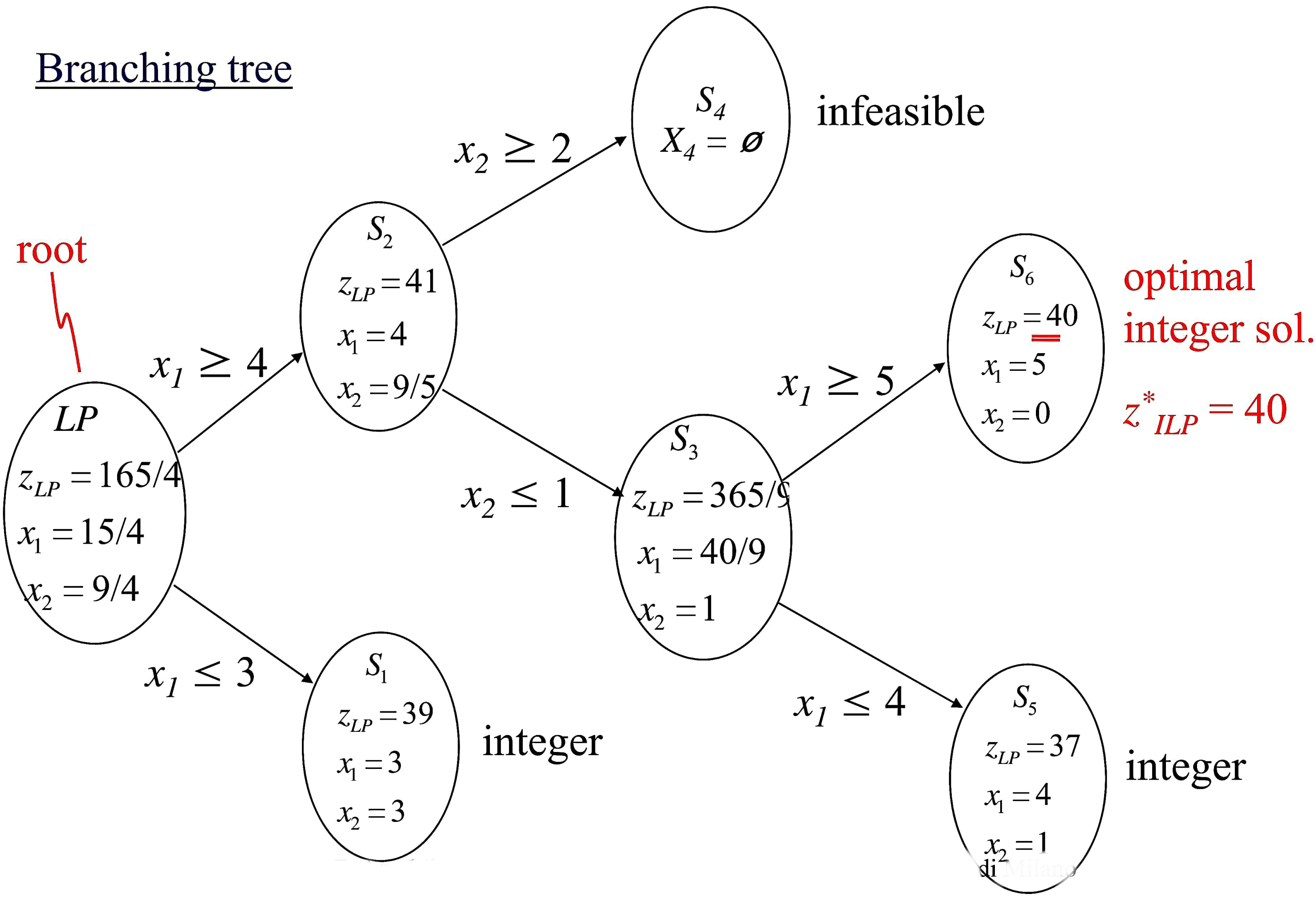
Integer solution $\overline{\underline{x}}_{LP5}$ (feasible for ILP) but with worse obj. fct. value of

$$\underline{x}' = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{with } z' = 39.$$

$\overline{\underline{x}}_{LP6}$ is the best integer solution found $\Rightarrow$ <u>optimal solution.</u>

|| Branch & Bound is an exact method (it guarantees an optimal solution).

Branching tree



The branching tree may not contain all possible nodes

(2ᵈ leaves)

A node of the tree has no child – is "fathomed"– if

- initial constraints + those on the arcs from the root are infeasible (e.g. S_4)
- optimal solution of the linear relaxation is integer (e.g. S_1)
- the value $\underline{c}^T\underline{\overline{x}}_{LP}$ of the optimal solution $\underline{\overline{x}}_{LP}$ of the linear relaxation is worse than that of the best feasible solution of ILP found so far.

≡ Bounding criterion

Observation: In the third case the feasible subregion of the subproblem associated to that node cannot contain an integer feasible solution that is better than the best feasible solution of ILP found so far!

Bounding criterion often allows to “discard” a number of nodes (subproblems).

Choice of the node (subproblem) to examine:

- First deeper nodes (depth-first search strategy)

  Simple recursive procedure, it is easy to reoptimize but it may be costly in case of wrong choice.

- First more promising nodes (best-bound first strategy)

  with the best value of linear relaxation

  Typically generates a smaller number of nodes but suproblems are less constrained ⇒ takes longer to find a first feasible solution and to improve it.

Choice of the (fractional) variable for branching

- It may not be the best choice to select the variable x_h whose fractional value is closer to 0,5 (hoping to obtain two subproblems that are more stringent and balanced).

- Strong branching: try to branch on some of candidate variables (fractional basic ones), evaluate the corresponding objective function values, and actually branch on the variable that yields the best improvement in the objective function.

Efficient solution of the linear relaxations

No need to solve the linear relaxations of the ILP subproblems from scratch with, for instance, the two-phase Simplex algorithm.

To efficiently find an optimal solution of the strengthened linear relaxation with a new constraint, we can exploit the optimal tableau of the previous linear relaxation and apply a single iteration of the **Dual simplex method** (variant of the Simplex method not covered here).

Applicability of Branch and Bound approach

Branch & Bound is also applicable to mixed ILPs:

when branching just consider the fractional variables that must be integer.

General method that can be adapted to tackle any discrete optimization problem and many nonlinear optimization problems.

e.g., scheduling, traveling salesman problem,…

We “just” need

- Technique to partition a set of feasible solutions into two or more subsets of feasible solutions (branch).
- Procedure to determine a bound on the cost of any solution in such a subset of feasible solutions (bound).

Observation: Branch-and-Bound can also be used as a heuristic by imposing an upper bound on the computing time or on the number of nodes that are examined.