

# SUBJECT NAME – OPTIMIZATION IN ENG. SUBJECT CODE – ROE6A001 LECTURE NOTES B.TECH 3<sup>rd</sup> YEAR – SEM- VI (2024-2025)



DEPARTMENT OF BASIC SCIENCE AND HUMANITIES

## **MODULE-I**

Engineering optimization involves finding the best solution to a problem, given certain constraints and objectives, by using mathematical models and algorithms. It's a systematic approach to design and improve various aspects of engineering systems, including resource allocation, scheduling, and process improvement. Essentially, it seeks to maximize or minimize a specific performance measure while adhering to constraints related to resources, materials, and operational conditions. Here's a more detailed breakdown.

1. Core Concepts:

- **Objective Function:** A real-valued function that defines what needs to be minimized or maximized (e.g., cost, efficiency, time).
- **Decision Variables:**Variables that can be changed by the designer to influence the outcome of the optimization process.
- **Constraints:**Limitations or restrictions that the solution must satisfy (e.g., resource availability, physical limitations).

2. Types of Optimization Problems:

- **Design Optimization:**Finding the best design among a set of alternatives, often using mathematical models of the design.
- **Resource Allocation:**Determining the optimal way to allocate resources (e.g., manpower, equipment) to achieve a desired outcome.
- **Scheduling:**Finding the best sequence of tasks to complete a project within a specific timeframe and constraints.
- **Process Improvement:**Identifying ways to optimize existing processes to improve efficiency, reduce costs, or enhance quality.

3. Methods for Solving Optimization Problems:

- **Deterministic Methods:**Algorithms that, given an initial solution, will always produce the same final solution.
- **Probabilistic Methods (Metaheuristics):**Algorithms that use random processes to explore the solution space and may not guarantee a global optimum, but can be faster for complex problems.
- **Metaheuristics:**Algorithms that use heuristic information to guide the search for a solution. Examples include genetic algorithms, simulated annealing, and particle swarm optimization.
- Linear Programming: A technique used to solve optimization problems with linear objective functions and constraints.
- **Non-linear Programming:** A technique used to solve optimization problems with non-linear objective functions and constraints.

4. Steps in the Optimization Process:

- 1. **Problem Definition:** Clearly define the objective, decision variables, and constraints.
- 2. **Model Development:** Create a mathematical model that accurately represents the optimization problem.

- 3. Algorithm Selection: Choose an appropriate optimization algorithm based on the problem's characteristics and complexity.
- 4. Solution Implementation: Use the chosen algorithm to find the optimal solution.
- 5. Validation and Refinement: Evaluate the solution's feasibility and effectiveness, and refine the model and algorithm as needed.
  - 5. Examples of Engineering Applications:
- **Structural Design:**Optimizing the dimensions of a bridge or building to minimize weight and cost while ensuring structural integrity.
- **Manufacturing:**Optimizing production processes to minimize costs, reduce waste, and improve product quality.
- **Energy Systems:**Optimizing the design and operation of power plants or energy grids to maximize efficiency and minimize environmental impact.
- **Supply Chain Management:**Optimizing logistics, inventory, and transportation to minimize costs and improve customer service.

## Optimization algorithms can be classified in several ways :

They can be categorized as deterministic or stochastic, gradient-based or gradient-free, first-order or second-order, and based on whether they tackle continuous or discrete optimization problems. Furthermore, they can be broadly divided into global and local optimization methods, with metaheuristics representing a problem-independent approach.

#### 1. Deterministic vs. Stochastic:

- **Deterministic algorithms:** follow a specific, repeatable sequence, leading to the same solution when started from the same initial point. Examples include hill-climbing and downhill simplex.
- **Stochastic algorithms:** incorporate randomness, potentially finding different solutions even with the same starting point. Genetic algorithms and hill-climbing with random restarts are examples.

## 2. Gradient-Based vs. Gradient-Free:

- **Gradient-based algorithms :**(also called derivative-based) utilize the gradient (derivative) information of the objective function to guide the search for an optimum. Steepest descent and Gauss-Newton methods are examples.
- **Gradient-free algorithms :**(also called derivative-free) do not rely on gradients and instead use only the objective function values. The Nelder-Mead downhill simplex method is a classic example.

## 3. First-Order vs. Second-Order:

- First-order algorithms: use the gradient (first derivative) to guide the search.
- **Second-order algorithms**: utilize the Hessian matrix (second derivative) to capture the curvature of the objective function, potentially leading to faster convergence.
  - 4. Continuous vs. Discrete Optimization:
- Continuous optimization: deals with problems where variables can take on any real value.
- **Discrete optimization**: addresses problems where variables are limited to a discrete set, often integers.

#### 5. Global vs. Local Optimization:

- Global optimization aims to find the absolute best solution, even if multiple local optima exist.
- Local optimization focuses on finding a good solution within a specific region, potentially getting stuck in a local optimum.

6. Metaheuristics:

- Metaheuristics: are problem-independent, general-purpose algorithms designed to tackle a wide range of optimization problems. They often employ strategies inspired by natural phenomena, such as evolution (genetic algorithms) or swarm intelligence (particle swarm optimization).
   Examples of Optimization Algorithms:
- **Genetic Algorithms:**Inspired by natural selection, these algorithms evolve a population of solutions over generations.
- **Particle Swarm Optimization (PSO):**Mimics the social behavior of bird flocks, with particles moving towards the best-known positions in the search space.
- **Simulated Annealing:**Inspired by the process of annealing in metallurgy, this algorithm gradually reduces the temperature of the search process, allowing it to escape local optima.
- Local Search Algorithms: Steepest descent, Newton's method, etc., are examples of algorithms that explore the neighborhood of the current solution to find a better one.



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## FORMULATION OF LINEAR PROGRAMMING PROBLEM

## INTRODUCTION TO LINEAR PROGRAMMING

Linear Programming is a problem solving approach that has been developed to help managers to make decisions.

Linear Programming is a mathematical technique for determining the optimum allocation of resources and obtaining a particular objective when there are alternative uses of the resources, money, manpower, material, machine and other facilities.

## THE NATURE OF LINEAR PROGRAMMING PROBLEM

Two of the most common are:

- 1. The product-mix problem
- 2. The blending Problem

In the product- mix problem there are two or more product also called candidates or activities competing for limited resources. The problem is to find out which products to include in production plan and in what quantities these should be produced or sold in order to maximize profit, market share or sales revenue.

The blending problem involves the determination of the best blend of available ingredients to form a certain quantity of a product under strict specifications. The best blend means the least cost blend of the required inputs.

## FORMULATION OF THE LINEAR PROGRAMMING MODEL

Three components are:

- 1. The decision variable
- 2. The environment (uncontrollable) parameters
- 3. The result (dependent) variable



Linear Programming Model is composed of the same components

## TERMINOLOGY USED IN LINEAR PROGRAMMING PROBLEM

**1. Components of LP Problem:** Every LPP is composed of a. Decision Variable, b. Objective Function, c. Constraints.

**2. Optimization:** Linear Programming attempts to either maximise or minimize the values of the objective function.

**3. Profit of Cost Coefficient:** The coefficient of the variable in the objective function express the rate at which the value of the objective function increases or decreases by including in the solution one unit of each of the decision variable.

**4.** Constraints: The maximisation (or minimisation) is performed subject to a set of constraints. Therefore LP can be defined as a constrained optimisation problem. They reflect the limitations of the resources.

**5. Input-Output coefficients:** The coefficient of constraint variables are called the Input-Output Coefficients. They indicate the rate at which a given resource is unitized or depleted. They appear on the left side of the constraints.

**6.** Capacities: The capacities or availability of the various resources are given on the right hand side of the constraints.

## THE MATHEMATICAL EXPRESSION OF THE LP MODEL

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The general LP Model can be expressed in mathematical terms as shown below:
Let
O<sub>ii</sub> = Input-Output Coefficient
C_i = Cost (Profit) Coefficient
b_i = Capacities (Right Hand Side)
X_i = Decision Variables
Find a vector (x_1, x_2, x_3, \dots, x_n) that minimise or maximise a linear objective function F(x)
where F(x) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n
subject to linear constraints
a_1x_1 + a_2x_2 + \dots + a_nx_n = b_2
a_1x_1 + a_2x_2 + \dots + a_nx_n \le b_2
.....
.....
.....
.....
am_1x_1 + am_2x_2 + \dots + am_nx_n \leq b_2
and non-negativity constraints
x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0
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## FORMULATION OF LPP STEPS

1. Identify decision variables

- 2. Write objective function
- 3. Formulate constraints

## **EXAMPLE 1. (PRODUCTION ALLOCATION PROBLEM)**

A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

	Time per unit (N	Machine Capacity		
Machine	Product 1	Product 2	Product 3	(minutes/day)
M1	2	3	2	440
$M_2$	4	-	3	470
M3	2	5	-	430

It is required to determine the daily number of units to be manufactured for each product. The profit per unit for product 1, 2 and 3 is Rs. 4, Rs.3 and Rs.6 respectively. It is assumed that all the amounts produced are consumed in the market. Formulate the mathematical (L.P.) model that will maximise the daily profit.

## Formulation of Linear Programming Model

## Step 1

From the study of the situation find the key-decision to be made. In the given situation key decision is to decide the extent of products 1, 2 and 3, as the extents are permitted to vary.

## Step 2

Assume symbols for variable quantities noticed in step 1. Let the extents (amounts) of products 1, 2 and 3 manufactured daily be  $x_1$ ,  $x_2$  and  $x_3$  units respectively.

## Step 3

Express the feasible alternatives mathematically in terms of variable. Feasible alternatives are those which are physically, economically and financially possible. In the given situation feasible alternatives are sets of values of  $x_1$ ,  $x_2$  and  $x_3$  units respectively.

where  $x_1, x_2 \text{ and } x_3 \ge 0.$ 

since negative production has no meaning and is not feasible.

## Step 4

Mention the objective function quantitatively and express it as a linear function of variables. In the present situation, objective is to maximize the profit.

i.e.,  $Z = 4x_1 + 3x_2 + 6x_3$ 

## Step 5

Put into words the influencing factors or constraints. These occur generally because of constraints on availability (resources) or requirements (demands). Express these constraints also as linear equations/inequalities in terms of variables.

Here, constraints are on the machine capacities and can be mathematically expressed as

$$\begin{array}{l} 2x_{1}\!\!+3x_{2}+2x_{3}\!\leq\!440\\ 4x_{1}\!\!+0x_{2}+3x_{3}\!\leq\!470\\ 2x_{1}\!\!+5x_{2}+0x_{3}\!\leq\!430 \end{array}$$

## **EXAMPLE 2: PRODUCT MIX PROBLEM**

A factory manufactures two products A and B. To manufacture one unit of A, 1.5 machine hours and 2.5 labour hours are required. To manufacture product B, 2.5 machine hours and 1.5 labour hours are required. In a month, 300 machine hours and 240 labour hours are available.

Profit per unit for A is Rs. 50 and for B is Rs. 40. Formulate as LPP.

## Solution:

Products	Resource/unit		
	Machine	Labour	
А	1.5	2.5	
В	2.5	1.5	
Availability	300 hrs	240 hrs	

There will be two constraints. One for machine hours availability and for labour hours availability.

Decision variables

 $X_1$  = Number of units of A manufactured per month.

 $X_2$  = Number of units of B manufactured per month.

The objective function:

Max  $Z = 50x_1 + 40x_2$ 

Subjective Constraints

For machine hours

 $1.5x_1 + 2.5x_2 \le 300$ 

For labour hours

 $2.5x_1 + 1.5x_2 \le 240$ 

Non negativity

 $x_1, x_2 \ge 0$ 

#### **EXAMPLE: 3**

A company produces three products A, B, C.

For manufacturing three raw materials P, Q and R are used.

Profit per unit

A - Rs. 5, B - Rs. 3, C - Rs. 4

Resource requirements/unit

Raw Material Product	Р	Q	R
A	-	20	50
В	20	30	-
С	30	20	40

Maximum raw material availability:

P - 80 units; Q - 100 units;

R - 150 units. Formulate LPP.

#### Solution:

Decision variables:

 $x_1 =$  Number of units of A

 $x_2 =$  Number of units of B

 $x_3 =$  Number of units of C

**Objective Function** 

Since Profit per unit is given, objective function is maximisation

Max  $Z = 5x_1 + 3x_2 + 4x_3$ 

#### **Constraints:**

For P:

For Q:

 $0x_1 + 20x_2 + 30x_3 \le 80$ 

 $20x_1 + 30x_2 + 20x_3 \le 100$ For R:

 $50x_1 + 0x_2 + 40x_3 \leq 150$ 

(for B, R is not required)

 $X_1, X_2, X_3 \ge 0$ 

#### **EXAMPLE 4: PORTFOLIO SELECTION (INVESTMENT DECISIONS)**

An investor is considering investing in two securities 'A' and 'B'. The risk and return associated with these securities is different.

Security 'A' gives a return of 9% and has a risk factor of 5 on a scale of zero to 10. Security 'B' gives return of 15% but has risk factor of 8.

Total amount to be invested is Rs. 5, 00, 000/- Total minimum returns on the investment should be 12%. Maximum combined risk should not be more than 6. Formulate as LPP. **Solution:** 

#### **Decision Variables:**

 $X_1$  = Amount invested in Security A  $X_2$  = Amount invested in Security B

#### **Objective Function:**

The objective is to maximise the return on total investment.

: Max Z = 0.09 X<sub>1</sub> + 0.015 X<sub>2</sub> ((% = 0.09, 15% = 0.15)

#### **Constraints:**

1. Related to Total Investment:  $X_1 + X_2 = 5, 00, 000$ 2. Related to Risk:  $5X_1 + 8X_2 = (6 X 5, 00, 000)$   $5X_1 + 8X_2 = 30, 00, 000$ 3. Related to Returns:  $0.09X_1 + 0.15X_2 = (0.12 X 5, 00, 000)$   $\therefore 0.09X_1 + 0.15X_2 = 60, 000$ 4. Non-negativity  $X_1, X_2 \ge 0$ 

#### **EXAMPLE 5: INSPECTION PROBLEM**

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1, 500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are Rs. 5 per hour while those of grade II inspector are Rs. 4 per hour. Any error made by an inspector costs Rs. 3 to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company, find the optimal assignment of inspectors that minimise the daily inspection cost.

#### Solution:

Let  $x_1$  and  $x_2$  denote the number of grade I and grade II inspectors that may be assigned the job of quality control inspection.

The objective is to minimise the daily cost of inspection. Now the company has to incur two types of costs; wages paid to the inspectors and the cost of their inspection errors. The cost of grade I inspector/hour is

Rs. (5 + 3 X 0.04 X 20) = Rs. 7.40.

Similarly, cost of grade II inspector/hour is

Rs.  $(4 + 3 \times 0.08 \times 14) =$  Rs. 7.36.

.:. The objective function is

minimise  $Z = 8(7.40x_1 + 7.36x_2) = 59.20 x_1 + 58.88x_2$ .

Constraints are

on the number of grade I inspectors:  $x_1 \le 10$ ,

on the number of grade II inspectors:  $x_2 \le 15$ on the number of pieces to be inspected daily:  $20 \ge 8x_1 + 14 \ge 8x_2 \ge 1500$ 

or  $160x_1 + 112x_2 \ge 1500$ 

#### where, $x_1, x_2 \ge 0$ .

## **EXAMPLE 6: TRIM LOSS PROBLEM**

A manufacturer of cylindrical containers receives tin sheets in widths of 30 cm and 60 cm respectively. For these containers the sheets are to be cut to three different widths of 15 cm, 21 cm and 27 cm respectively. The number of containers to be manufactured from these three widths are 400, 200 and 300 respectively. The bottom plates and top covers of the containers are purchased directly from the market. There is no limit on the lengths of standard tin sheets. Formulate the LPP for the production schedule that minimises the trim losses.

#### Solution:

Key decision is to determine how each of the two standard widths of tin sheets be cut to the require widths so that trim losses are minimum.

From the available widths of 30 cm and 60 cm, several combinations of the three required widths of 15 cm, 21 cm and 27 cm are possible.

Let x<sub>ij</sub> represent these combinations. Each combination results in certain trim loss.

Constraints can be formulated as follows:

The possible cutting combinations (plans) for both types of sheets are shown in the table below:

Width (cm)	i = I (30	) cm)		i = II (60	) cm)				
	X11	X <sub>12</sub>	X <sub>13</sub>	X <sub>21</sub>	X <sub>22</sub>	X <sub>23</sub>	X <sub>24</sub>	X <sub>25</sub>	X <sub>26</sub>
15	2	0	0	4	2	2	1	0	0
21	0	1	0	0	1	0	2	1	0
27	0	0	1	0	0	1	0	1	2
Trim Loss	0	9	3	0	9	3	3	12	6
(cm)									

Thus, the constraints are

 $2x_{11} + 4x_{21} + 2x_{22} + 2x_{23} + x_{24} \ge 400$ 

 $x_{12} \ + x_{22} + \ 2x_{24} + x_{25} \ \geq 200$ 

 $x_{13} + x_{23} + x_{25} + x_{26} \geq 300$ 

Objective is to maximise the trim losses.

i.e., minimise  $Z = 9x_{12} + 3x_{13} + 9x_{22} + 3x_{23} + 3x_{24} + 12x_{25} + 6x_{26}$ 

where  $x_{11}$ ,  $x_{12}$ ,  $x_{13}$ ,  $x_{21}$ ,  $x_{22}$ ,  $x_{23}$ ,  $x_{24}$ ,  $x_{25}$ ,  $x_{26} \ge 0$ .

#### **EXAMPLE 7: MEDIA SELECTION**

An advertising agency is planning to launch an ad campaign. Media under consideration are T.V., Radio & Newspaper. Each medium has different reach potential and different cost.

Minimum 10, 000, 000 households are to be reached through T.V. Expenditure on newspapers should not be more than Rs. 10, 00, 000. Total advertising budget is Rs. 20 million.

Following data is available:

Medium	Cost per Unit (Rs.)	Reach per unit (No. of households)
Television	2, 00, 000	20, 00, 000

Radio	80, 000	10, 00, 000
Newspaper	40, 000	2, 00, 000

#### Solution:

**Decision Variables:** 

	$x_1 =$ Number of units of T.V	. ads,		
	$x_2 =$ Number of units of Rac	lio ads,		
	$x_3 =$ Number of units of New	wspaper ads.		
Objective function: (	Maximise reach)			
Max.	$Z = 20, 00, 000 x_1 + 10, 00, 0$	$00 x_2 + 2, 00, 000 x_3$		
Subject to constraints	3:			
	20, 00, 000 $x_1 \ge 10, 000, 000$	) (for T.V.)		
	40, 000 $x_3 \le 10, 00, 000$	(for Newspaper)		
2, 00, $000x_1 + 80, 00$	$0x_2 + 40,000x_3 \leq 20,000,000$	(Ad. budget)		
$x_1, x_2, x_3 \ge 0$				
: Simplifying constr	aints:			
for T.V.	$2 x_1 \ge 10$	$\therefore x_1 \ge 5$		
for Newspaper	$4 x_3 \le 100$	$\therefore x_3 \leq 25$		
Ad. Budget				
$20 x_1 + 8 x_2 + 4 x_3 \le 2000$				
$5 x_1 + 2x_2 + x_3 \le 500$				
	$x_1, x_2, x_3 \ge 0$			

#### **EXAMPLE 8: DIET PROBLEM**

Vitamins  $B_1$  and  $B_2$  are found in two foods  $F_1$  and  $F_2$ . 1 unit of  $F_1$  contains 3 units of  $B_1$  and 4 units of  $B_2$ . 1 unit of  $F_2$  contains 5 units of  $B_1$  and 3 units of  $B_2$  respectively.

Minimum daily prescribed consumption of  $B_1 \& B_2$  is 50 and 60 units respectively. Cost per unit of  $F_1 \& F_2$  is Rs. 6 & Rs. 3 respectively.

Formulate as LPP.

#### Solution:

Vitamins	Foo	Minimum	
	$F_1$	$F_2$	Consumption
B1	3	5	30
$B_2$	5	7	40

**Decision Variables:** 

 $x_1 = No.$  of units of  $P_1$  per day.

 $x_2 = No.$  of units of  $P_2$  per day.

Objective function:

Min.  $Z = 100 x_1 + 150 x_2$ 

Subject to constraints:

 $3x_1 + 5x_2 \ge 30$  (for N<sub>1</sub>)  $5x_1 + 7x_2 \ge 40$  (for N<sub>2</sub>)  $x_1, x_2 \ge 0$ 

#### **EXAMPLE 9: BLENDING PROBLEM**

A manager at an oil company wants to find optimal mix of two blending processes. Formulate LPP.

Data:

Process	Input (Cr	rude Oil)	Output (0	Gasoline)	
	Grade A	Grade B	Х	Y	
P1	6	4	6	9	
P <sub>2</sub>	5	6	5	5	
Profit per operatio	on: Process	$1 (P_1) = Rs. 4, 000$	0		
	Process	$2(P_2) = Rs. 5,000$	0		
Maximum availab	ility of crude oil:	Grade $A = 3$	500 units		
		Grade $B = 4$	400 units		
Minimum Deman	d for Gasoline:	X = 300  un	X = 300 units		
		Y = 200  un	its		
Solution:					
Decision Variable	s:				
$x_1 = No.$ of operations of $P_1$					
$x_2 = No.$ of operations of $P_2$					
Objective Functio	n:	-			
Max. $Z = 4000 x_1 + 5000 x_2$					
Subjective to cons	traints:				
$6x_{1} + 5x_{2} \le 500$					

 $6x_{1} + 5x_{2} \le 500$   $4x_{1} + 6x_{2} \le 400$   $6x_{1} + 5x_{2} \ge 300$   $9x_{1} + 5x_{2} \ge 200$   $x_{1}, x_{2} \ge 0$ 

#### **EXAMPLE 10: FARM PLANNING**

A farmer has 200 acres of land. He produces three products X, Y & Z. Average yield per acre for X, Y & Z is 4000, 6000 and 2000 kg.

Selling price of X, Y & Z is Rs. 2, 1.5 & 4 per kg respectively. Each product needs fertilizers. Cost of fertilizer is Rs. 1 per kg. Per acre need for fertilizer for X, Y & Z is 200, 200 & 100 kg respectively. Labour requirements for X, Y & Z is 10, 12 & 10 man hours per acre. Cost of labour is Rs. 40 per man hour. Maximum availability of labour is 20, 000 man hours. Formulate as LPP to maximise profit.

#### Solution:

Decision variables:

The production/yield of three products X, Y & Z is given as per acre. Hence,

 $x_1 = No.$  of acres allocated to X

 $x_2 = No.$  of acres allocated to Y

 $x_3 = No.$  of acres allocated to Z

**Objective Function:** 

Profit = Revenue - Cost

Profit = Revenue - (Fertiliser Cost + Labour Cost)

Product	Х	Y	Z
Revenue	2 (4000) x <sub>1</sub>	1.5 (6000) x <sub>2</sub>	4 (2000) x <sub>3</sub>
(-) Less:			

Fertiliser Cost	1 (200) x <sub>1</sub>	1 (200) x <sub>2</sub>	1 (100) x <sub>3</sub>
Labour Cost	40 (10) x <sub>1</sub>	40 (12) x <sub>2</sub>	40 (10) x <sub>3</sub>
Profit	7400 x <sub>1</sub>	8320 x <sub>2</sub>	7500 x <sub>3</sub>

: Objective function

Max. =  $7400 x_1 + 8320 x_2 + 7500 x_3$ 

Subject to constraints:

 $\begin{array}{ll} x_1 + \ x_2 + \ x_3 = 200 & (\mbox{Total Land}) \\ 10 \ x_1 + 12 \ x_2 + 10 \ x_3 \leq 20, \ 000 & (\mbox{Max Man hours}) \\ x_1, \ x_2, \ x_3 \geq 0 & \end{array}$ 

## **MERITS OF LPP**

- 1. Helps management to make efficient use of resources.
- 2. Provides quality in decision making.
- 3. Excellent tools for adjusting to meet changing demands.
- 4. Fast determination of the solution if a computer is used.
- 5. Provides a natural sensitivity analysis.
- 6. Finds solution to problems with a very large or infinite number of possible solution.

#### **DEMERITS OF LPP**

**1. Existence of non-linear equation:** The primary requirements of Linear Programming is the objective function and constraint function should be linear. Practically linear relationship do not exist in all cases.

**2. Interaction between variables:** LP fails in a situation where non-linearity in the equation emerge due to joint interaction between some of the activities like total effectiveness.

3. Fractional Value: In LPP fractional values are permitted for the decision variable.

**4. Knowledge of Coefficients of the equation:** It may not be possible to state all coefficients in the objective function and constraints with certainty.

#### **EXERCISES**

1. Explain what is meant by decision variables, objective function and constraints in Linear Programming.

2. Give the mathematical formulation of the linear programming problems.

3. What are the components of LPP? What is the significance of non-negativity restriction?

4. State the limitations of LPP.

5. Give the assumptions and advantages of LPP.

6. An investor wants to identify how much to invest in two funds, one equity and one debt. Total amount available is Rs. 5, 00, 000. Not more than Rs. 3, 00, 000 should be invested in a single fund. Returns expected are 30% in equity and 8% in debt. Minimum return on total investment should be 15%. Formulate as LPP.

7. A company manufactures two products  $P_1$  and  $P_2$ . Profit per unit for  $P_1$  is Rs. 200 and for  $P_2$  is Rs. 300. Three raw materials  $M_1$ ,  $M_2$  and  $M_3$  are required. One unit of  $P_1$  needs 5 units of  $M_1$  and 10 units of  $M_2$ . One unit of  $P_2$  needs 18 units of  $M_2$  and 10 units of  $M_3$ . Availability is 50 units of  $M_1$ , 90 units of  $M_2$  and 50 units of  $M_3$ . Formulate as LPP.

8. A firm produces two products X and Y. Minimum 50 units of X should be produced. There is no limit for producing Y. Profit per unit is Rs. 100 for X and Rs. 150 for Y.

Product	<b>Resource Requirement</b>	Resource Availability
X	20 Machine Hours	Machine Hours $= 2500$
	10 Labour Hours	Labour Hours $= 3000$
Y	10 Machine Hours	
	15 Labour Hours	

Formulate as LPP.

9. A patient has been recommended two nutrients  $N_1$  and  $N_2$  everyday. Minimum intake is 10g for  $N_1$  and 15g for  $N_2$  everyday.

These nutrients are available in two products  $P_1$  and  $P_2$ . One unit of  $P_1$  contains 2g of  $N_1$  and 3g of  $N_2$ . One unit of  $P_2$  contains 1g of  $N_1$  and 2g of  $N_2$ . Cost per unit is Rs. 200 for  $P_1$  and Rs. 150 for  $P_2$ .

Formulate as LPP such that nutrient requirement can be fulfilled at the lowest cost.

10. Two vitamins A and B are to be given as health supplements on daily basis to students. There are two products Alpha & Beta which contain vitamins A and B. One unit of Alpha contains 2g of A and 1g of B. One unit of Beta contains 1g of A and 2g of B. Daily requirements for A and B are atleast 10g each. Cost per unit of Alpha is Rs. 20 and of Beta is Rs. 30. Formulate as LPP to satisfy the requirements at minimum cost.

# LINEAR PROGRAMMING SOLUTION - GRAPHICAL METHOD INTRODUCTION

There are two methods available to find optimal solution to a Linear Programming Problem. One is graphical method and the other is simplex method.

Graphical method can be used only for a two variables problem i.e. a problem which involves two decision variables. The two axes of the graph (X & Y axis) represent the two decision variables  $X_1 \& X_2$ .

## METHODOLOGY OF GRAPHICAL METHOD

#### Step 1: Formulation of LPP (Linear Programming Problem)

Use the given data to formulate the LPP.

#### Maximisation

#### Example 1

A company manufactures two products A and B. Both products are processed on two machines  $M_1 \& M_2$ .

	M1	M <sub>2</sub>
А	6 Hrs/Unit	2 Hrs/Unit
В	4 Hrs/Unit	4 Hrs/Unit
Availability	7200 Hrs/month	4000 Hrs/month

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out the monthly production of A and B to maximise profit by graphical method.

Formulation of LPP

 $X_1 = No. of units of A/Month$ 

 $X_2 = No.$  of units of B/Month

 $Max \ Z = 100 \ X_1 + 80 \ X_2$ 

Subject to constraints:

 $6 X_1 + 4 X_2 \le 7200$ 

 $2 X_1 + 4 X_2 \le 4000$ 

 $X_{1,}X_{2} \geq 0$ 

#### Step 2: Determination of each axis

Horizontal (X) axis: Product A (X<sub>1</sub>)

Vertical (Y) axis: Product B (X<sub>2</sub>)

Step 3: Finding co-ordinates of constraint lines to represent constraint lines on the graph.

The constraints are presently in the form of inequality ( $\leq$ ). We should convert them into equality to obtain co-ordinates.

#### Constraint No. 1: 6 $X_1 + 4 X_2 \le 7200$

Converting into equality:

 $6 \; X_1 + 4 \; X_2 \leq 7200$ 

 $X_1$  is the intercept on X axis and  $X_2$  is the intercept on Y axis.

To find  $X_1$ , let  $X_2 = 0$ 

 $6 X_1 = 7200$ 

 $6 X_1 = 7200$  $\therefore X_1 = 1200; X_2 = 0 (1200, 0)$ To find  $X_2$ , let  $X_1 = 0$  $X_2 = 7200; X_1 = 0 (0, 1800)$ Hence the two points which make the constraint line are:

(1200, 0) and (0, 1800)

Note: When we write co-ordinates of any point, we always write  $(X_1, X_2)$ . The value of  $X_1$  is written first and then value of  $X_2$ . Hence, if for a point  $X_1$  is 1200 and  $X_2$  is zero, then its co-ordinates will be (1200, 0).

Similarly, for second point,  $X_1$  is 0 and  $X_2$  is 1800. Hence, its co-ordinates are (0, 1800).

**Constraint No. 2:** 

Each constraint will be represented by a single straight line on the graph. There are two constraints, hence there will be two straight lines.

The co-ordinates of points are:

1. Constraint No. 1:	(1200, 0) and (0, 1800)
2. Constraint No. 2:	(2000, 0) and (0, 1000)

## Step 4: Representing constraint lines on graph

To mark the points on the graph, we need to select appropriate scale. Which scale to take will depend on maximum value of  $X_1 \& X_2$  from co-ordinates.

For  $X_1$ , we have 2 value  $\rightarrow$  1200 and 2000

 $\therefore$ Max. value for X<sub>2</sub> = 2000

For  $X_2$ , we have 2 value  $\rightarrow$  1800 and 1000

 $\therefore$ Max. value for X<sub>2</sub> = 1800

Assuming that we have a graph paper 20 X 30 cm. We need to accommodate our lines such that for X-axis, maximum value of 2000 contains in 20 cm.

 $\therefore$  Scale 1 cm = 200 units

 $\therefore 2000 \text{ units} = 10 \text{ cm} \qquad (X-axis)$ 1800 units = 9 cm (Y-axis)

The scale should be such that the diagram should not be too small.

## **Constraint No. 1:**

The line joining the two points (1200, 0) and (0, 1800) represents the constraint 6  $X_1 + 4 X_2 \le 7200$ .



(1200, 0)

Every point on the line will satisfy the equation (equality)  $6 X_1 + 4 X_2 \le 7200$ . Every point below the line will satisfy the inequality (less than)  $6 X_1 + 4 X_2 \le 7200$ .

#### **Constraint No. 2:**

The line joining the two points (2000, 0) and (0, 1000) represents the constraint  $2 X_1 + 4 X_2 \le 4000$ 

Every point on the line will satisfy the equation (equality)  $2 X_1 + 4 X_2 \le 4000$ . Every point below the line will satisfy the inequality (less than)  $2 X_1 + 4 X_2 \le 4000$ .



Now the final graph will look like this:



#### **Step 5: Identification of Feasible Region**

The feasible region is the region bounded by constraint lines. All points inside the feasible region or on the boundary of the feasible region or at the corner of the feasible region satisfy all constraints.

Both the constraints are 'less than or equal to' ( $\leq$ ) type. Hence, the feasible region should be inside both constraint lines.

Hence, the feasible region is the polygon OABC. 'O' is the origin whose coordinates are (0, 0). O, A, B and C are called vertices of the feasible region.



#### **Step 6: Finding the optimal Solution**

The optimal solution always lies at one of the vertices or corners of the feasible region. To find optimal solution:

We use corner point method. We find coordinates  $(X_1, X_2 \text{ Values})$  for each vertex or corner point. From this we fine 'Z' value for each corner point.

Vertex	Co-ordinates	$Z = 100 X_1 + 80 X_2$
0	$X_1 = 0, X_2 = 0$	$\mathbf{Z} = 0$
	From Graph	
А	$X_1 = 0, X_2 = 1000$	Z = Rs. 80, 000
	From Graph	

В	$X_1 = 800, X_2 = 600$	Z = Rs. 1, 28, 000
	From Simultaneous equations	
С	$X_1 = 1200, X_2 = 0$	Z = Rs. 1, 20, 000
	From Graph	

Max. Z = Rs. 1, 28, 000 (At point B)

For B  $\longrightarrow$  B is at the intersection of two constraint lines 6 X<sub>1</sub> + 4 X<sub>2</sub>  $\leq$  7200 and 2 X<sub>1</sub> + 4

(1)

(2)

 $X_2 \leq 4000$ . Hence, values of  $X_1$  and  $X_2$  at B must satisfy both the equations.

We have two equations and two unknowns,  $X_1$  and  $X_2$ . Solving simultaneously.

 $6 X_1 + 4 X_2 \le 7200$ 

 $2 X_1 + 4 X_2 \le 4000$   $4 X_1 = 3200$   $X_1 = 800$ Subtracting (2) from (1)

Substituting value of  $X_1$  in equation (1), we get

 $4 X_2 = 2400$   $\therefore X_2 = 600$ 

Solution

Optimal Profit = Max Z = Rs. 1, 28, 000 Product Mix:

 $X_1 = No. of units of A / Month = 800$ 

 $X_2 = No. of units of A / Month = 600$ 

#### **ISO Profit line:**

ISO profit line is the line which passes through the points of optimal solution (Maximum Profit). The slope of the iso-profit line depends on the objective function.

In the above example, the objective function is:

Max.  $Z = 100 X_1 + 80 X_2$ 

How to find slope of iso-profit line:

Equation of a straight line: y = mx + c

where, m = slope of the straight line

In our case, y means ' $X_2$ ' and x means ' $X_1$ '.

c means 'Z'.

 $\therefore X_2 = m \cdot X_1 + Z$ 

Converting original objective function in this format:

Max.  $Z = 100 X_1 + 80 X_2$ 

 $\therefore 80 X_2 = Z - 100 X_1 = -100 X_1 + Z$ 

$$\therefore X_2 = \frac{-100}{80} X_1 + \frac{Z}{80}$$
$$\therefore X_2 = \frac{-5}{4} X_1 + \frac{Z}{80}$$

 $\therefore$  Slope of ISO profit line =  $\frac{-5}{4}$ 

Negative sign indicates that the line will slope from left to right downwards. And slope will be 5/4. Every 4 units on X-axis for 5 units on Y-axis.



As we start from the origin and go away from origin to maximise profit, 'B' is the last point on the feasible region that is intersected by the iso-profit line. Hence, B is the optimal solution.

## MINIMISATION

## Example 2

A firm is engaged in animal breeding. The animals are to be given nutrition supplements everyday. There are two products A and B which contain the three required nutrients.

Nutrients	Quantity/unit		Minimum Requirement
	Α	B	
1	72	12	216
2	6	24	72
3	40	20	200

Product cost per unit are: A: rs. 40; B: Rs. 80. Find out quantity of product A & B to be given to provide minimum nutritional requirement.

Step 1: Formulation as LPP  $X_1$  - Number of units of A X<sub>2</sub> - Number of units of B Z - Total Cost Min.  $Z = 40 X_1 + 80 X_2$ Subject to constraints:  $72 X_1 + 12 X_2 \ge 216$  $6 X_1 + 24 X_2 \ge 72$  $40 X_1 + 20 X_2 \ge 200$  $X_1, X_2 \ge 0.$ Step 2: Determination of each axis Horizontal (X) axis: Product A  $(X_1)$ Vertical (Y) axis: Product B (X<sub>2</sub>) Step 3: Finding co-ordinates of constraint lines to represent the graph All constraints are 'greater than or equal to' type. We should convert them into equality: **1. Constraint No. 1:**  $72 X_1 + 12 X_2 \ge 216$ Converting into equality

1. Constraint No. 1: (3, 0) & (0, 18)		
The co-ordinates of points are:		
$20 X_2 = 200$	$\therefore X_1 = 0, X_2 = 10$	(0, 10)
To find $X_2$ , let $X_1 = 0$		
$40 X_1 = 200$	$\therefore X_1 = 5, X_2 = 0$	(5, 0)
To find $X_1$ , let $X_2 = 0$		
$40 X_1 + 20 X_2 \ge 200$		
3. Constraint No. 3:		
$24 X_2 = 72$	$\therefore X_1 = 0, X_2 = 3$	(0, 3)
To find $X_2$ , let $X_1 = 0$		
$6 X_1 = 72$	$\therefore X_1 = 12, X_2 = 0$	(12, 0)
To find $X_1$ , let $X_2 = 0$		
$6 X_1 + 24 X_2 \ge 72$		
2. Constraint No. 2:		
$12 X_2 = 216$	$\therefore X_1 = 0, X_2 = 18$	(0, 18)
To find $X_2$ , let $X_1 = 0$		
$72 X_1 = 216$	$\therefore X_1 = 3, X_2 = 0$	(3, 0)
To find $X_1$ , let $X_2 = 0$		
$72 X_1 + 12 X_2 = 216$		

2. Constraint No. 2: (12, 0) & (0, 3)

3. Constraint No. 3: (5, 0) & (0, 5)

Every point on the line will satisfy the equation (equality) 72  $X_1 + 12 X_2 = 216$ . Every point above the line will satisfy the inequality (greater than) 72  $X_1 + 12 X_2 = 216$ .

Similarly, we can draw lines for other two constraints.

**Step 5: Feasible Region** 



All constraints are greater than or equal to  $(\geq)$  type. Hence, feasible region should be above (to the right of) all constraints.

The vertices of the feasible region are A, B, C & D.

# **Step 6: Finding the optimal solution**

Conner I onne Miethiod		
Vertex	Co-ordinates	$Z = 40 X_1 + 80 X_2$
А	$X_1 = 0, X_2 = 18$	$\therefore$ Z = 1, 440
	From Graph	
В	$X_1 = 2, X_2 = 6$	$\therefore Z = 560$
	From Simultaneous Equations	
C	$X_1 = 4, X_2 = 2$	$\therefore$ Z = 320
	From Simultaneous Equations	<b>7</b> 100
D	$X_1 = 12, X_2 = 0$	$\therefore$ Z = 480
. M: 7 D. 200 (At a	From graph	
$\therefore$ Min. $Z = Ks. 320$ (At p	oint C)	$\Delta X > \Delta 1 c = 1.140 X = 0.0 X$
For B - Point B is at inte	rsection of constraint lines $72 X_1 + 1$	$2 X_2 \ge 216$ and $40 X_1 + 20 X_2$
$\geq$ 200'. Hence, point B sl	hould satisfy both the equations.	
$72 X_1 + 12 X_2 = 216$	(1)	
$40 X_1 + 20 X_2 = 200$	(2)	
$\therefore 360 X_1 + 60 X_2 = 1080$	(1) x 5	
$120 X_1 + 60 X_2 = 600$	(2) x 3	
$\therefore 240 X_1 = 480$		
$X_1 = 2$		
Substituting value of $X_1$	in equation (1), we get:	
$12 X_2 = 216 - 144 = 72$		
$X_2 = 6$		
For C - Point C is at inte	ersection of constraint lines '6 $X_1 + 2$	$4 X_2 = 72'$ and $40 X_1 + 20 X_2 = -1000$
For C - Point C is at inte 200'. Hence, point C sho	rsection of constraint lines '6 $X_1$ + 2 uld satisfy both the equations	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> =
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$	Prsection of constraint lines '6 $X_1$ + 2 uld satisfy both the equations.	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> = (1)
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$	Prsection of constraint lines '6 $X_1$ + 2 uld satisfy both the equations.	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> = (1) (2)
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$	The section of constraint lines '6 $X_1$ + 2 and a satisfy both the equations.	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> = (1) (2) (1) x 5
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$	The section of constraint lines '6 $X_1$ + 2 and a satisfy both the equations.	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$	The section of constraint lines '6 $X_1$ + 2 and a satisfy both the equations.	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$	The section of constraint lines '6 $X_1$ + 2 and a satisfy both the equations.	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$	The equation of constraint lines '6 $X_1$ + 2 and a satisfy both the equations.	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of X <sub>1</sub> if	The equation of constraint lines '6 $X_1$ + 2 and a satisfy both the equations. The equation (1), we get	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ is $24 X_2 = 72 - 24 = 48$	The equation of constraint lines '6 $X_1$ + 2 and a satisfy both the equations. In equation (1), we get	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$	The equation of constraint lines '6 $X_1$ + 2 and satisfy both the equations. The equation (1), we get	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ Solution	The equation of constraint lines '6 $X_1$ + 2 and satisfy both the equations. In equation (1), we get	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ Solution Optimal Cost = Z min = 1	The equation of constraint lines '6 $X_1$ + 2 and satisfy both the equations. The equation (1), we get Rs. 320	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ <b>Solution</b> Optimal Cost = Z min = 1 Optimal Product Mix:	resection of constraint lines '6 $X_1$ + 2 uld satisfy both the equations. In equation (1), we get Rs. 320	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ <b>Solution</b> Optimal Cost = Z min = 1 Optimal Product Mix: $X_1 = N$	The equation of constraint lines '6 $X_1$ + 2 and satisfy both the equations. In equation (1), we get Rs. 320 No. of units of product A = 4	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ <b>Solution</b> Optimal Cost = Z min = 1 Optimal Product Mix: $X_1 = N$ $X_2 = N$	The equation of constraint lines '6 $X_1$ + 2 and satisfy both the equations. In equation (1), we get Rs. 320 No. of units of product A = 4 No. of units of product B = 2	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ Solution Optimal Cost = Z min = 1 Optimal Product Mix: $X_1 = N$ $X_2 = N$ ISO Cost Line	The equation of constraint lines '6 $X_1$ + 2 and satisfy both the equations. The equation (1), we get Rs. 320 No. of units of product A = 4 No. of units of product B = 2	4 $X_2$ = 72' and '40 $X_1$ + 20 $X_2$ = (1) (2) (1) x 5 (2) x 6
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ <b>Solution</b> Optimal Cost = Z min = 1 Optimal Product Mix: $X_1 = N$ $X_2 = N$ <b>ISO Cost Line</b> ISO Cost line passes throw	The point of product $A = 4$ No. of units of product $A = 2$ The point of optimal solution (Mission 2004) The point of optimal solution (Mission 2004)	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> = (1) (2) (1) x 5 (2) x 6 inimum cost)
For C - Point C is at inte 200'. Hence, point C show $6 X_1 + 24 X_2 = 72$ $40 X_1 + 20 X_2 = 200$ $30 X_1 + 120 X_2 = 360$ $240 X_1 + 120 X_2 = 1200$ $210 X_1 = 840$ $X_1 = 4$ Substituting value of $X_1$ if $24 X_2 = 72 - 24 = 48$ $X_2 = 2$ <b>Solution</b> Optimal Cost = Z min = 1 Optimal Product Mix: $X_1 = N$ $X_2 = N$ <b>ISO Cost Line</b> ISO Cost line passes throw Objective function: Z = 4	The point of product $A = 4$ No. of units of product $A = 4$ No. of units of product $B = 2$ The point of optimal solution (Minimum 2005) No. $X_1 + 80 X_2$	4 X <sub>2</sub> = 72' and '40 X <sub>1</sub> + 20 X <sub>2</sub> = (1) (2) (1) x 5 (2) x 6 inimum cost)

where m = slope

In this case, y is  $X_2 \& x$  is  $X_1$   $Z = 40 X_1 + 80 X_2$   $\therefore 80 X_2 = -40 X_1 + Z$   $\therefore X_2 = -1/2x_1 + Z/80$   $\therefore$  Slope of ISO-cost line = -1/2 Sloping from left to right downwards

 $\therefore$  Point C is the nearest to the origin.

# MAXIMISATION-MIXED CONSTRAINTS

#### Example 1

A firm makes two products  $P_1 \& P_2$  and has production capacity of 18 tonnes per day.  $P_1 \& P_2$  require same production capacity. The firm must supply at least 4 t of  $P_1 \& 6$  t of  $P_2$  per day. Each tonne of  $P_1 \& P_2$  requires 60 hours of machine work each. Maximum machine hours available are 720. Profit per tonne for  $P_1$  is Rs. 160 &  $P_2$  is Rs. 240. Find optimal solution by graphical method.

Direction for minimisation

#### **LPP** Formulation

 $X_1 = \text{Tonnes of } P_1 / \text{Day}$  $X_2 = Tonnes of P_2 / Day$ Max.  $Z = 160 X_1 + 240 X_2$ Subject to constraints  $X_1 \ge 4$  $X_2 \ge 6$  $X_1 + X_2 \le 18$  $60 X_1 + 60 X_2 \le 720$  $X_{1}, X_{2} \ge 0$ Coordinates for constraint lines: 1.  $X_1 \ge 4$ (4, 0) .... No value for  $X_2, \therefore X_2 = 0$ 2.  $X_2 \ge 6$ (0, 6) .... No value for  $X_1, \therefore X_1 = 0$ 3.  $X_1 + X_2 \le 18$ (18, 0) (0, 18)4. 60 X<sub>1</sub> + 60 X<sub>2</sub>  $\leq$  720 (12, 0) (0, 12)If  $X_1 = 0$ , 60  $X_2 = 720$  $\therefore X_2 = 12 (0, 12)$ 

If  $X_2 = 0, 60 X_1 = 720$ Graph:  $X_1$ : X Axis  $X_2$ : Y Axis

Scale:

Maximum value for  $X_1 = 18$ ; Maximum value for  $X_2 = 18$ ;  $\therefore$  Scale: 1 cm = 2 Tonnes.



Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABC.

**Optimal Solution** 

Corner Point Method

Vertex	Coordinates	$Z = 160 X_1 + 240 X_2$
А	$X_1 = 4, X_2 = 8$	$\therefore$ Z = Rs. 2, 560
	Simultaneous Equation	
В	$X_1 = 4, X_2 = 6$	$\therefore$ Z = Rs. 2, 080
	From Graph	
С	$X_1 = 6, X_2 = 6$	$\therefore$ Z = Rs. 2, 400
	Simultaneous Equations	

For A  $\rightarrow$  X<sub>1</sub> = 4 from graph A is on the line 60 X<sub>1</sub> + 60 X<sub>2</sub> = 720 60 X<sub>2</sub> = 720 - 60 (4) = 480  $\therefore$  X<sub>2</sub> = 8 For C  $\rightarrow$  X<sub>2</sub> = 6 from graph A is on the line 60 X<sub>1</sub> + 60 X<sub>2</sub> = 720 60 X<sub>1</sub> = 720 - 60 (6) = 360  $\therefore$  X<sub>1</sub> = 6  $\therefore$  Z = Rs. 2, 560 (At point 'A') **Solution** Optimal Profit Z. Max = Rs. 2, 560 X<sub>1</sub> = Tonnes of P<sub>1</sub> = 4 tonnes X<sub>2</sub> = Tonnes of P<sub>2</sub> = 8 tonnes.

#### MINIMISATION MIXED CONSTRAINTS

Example 1:

A firm produces two products P and Q. Daily production upper limit is 600 units for total production. But at least 300 total units must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1200 machine hours must be used daily. Manufacturing costs per unit are Rs. 50 for P and Rs. 20 for Q. Find optimal solution for the LPP graphically.

LPP formulation  $X_1 = No. of Units of P / Day$  $X_2 = No. of Units of Q / Day$ Min.  $Z = 50 X_1 + 20 X_2$ Subject to constraints  $X_1 + X_2 \le 600$  $X_1 + X_2 \ge 300$  $6 X_1 + 2 X_2 \ge 1200$  $X_1, X_2 \ge 0$ Coordinates for Constraint lines 1.  $X_1 + X_2 = 600$ If  $X_1 = 0$ ,  $X_2 = 600$ (0, 600)If  $X_2 = 0$ , 60  $X_1 = 600$ : (600, 0) 2.  $X_1 + X_2 = 300$ If  $X_1 = 0$ ,  $X_2 = 300$ ∴ (0, 300) If  $X_2 = 0$ , 60  $X_1 = 300$ : (300, 0) 3.  $6 X_1 + 2 X_2 \ge 1200$ If  $X_1 = 0$ ,  $2X_2 = 1200$  $\therefore X_2 = 600$ (0, 600)If  $X_2 = 0$ , 6  $X_1 = 1200$  $\therefore X_1 = 200$ (200, 0)Graph: X1: X Axis X<sub>2</sub>: Y Axis

Scale:

Maximum value for  $X_1 = 600$ ; Maximum value for  $X_2 = 600$ ;  $\therefore$  Scale: 1 cm = 50 units. Feasible region is ABCD.



Two constraints are 'greater than or equal to' type. Hence, feasible region will be above or to the right of these constraint lines. Two constraints are 'less than or equal to' type. Hence, feasible region will be below or to the left of these constraint lines. Hence, feasible region is ABCD.

## **Optimal Solution Corner Point Method**

Vertex	Coordinates	$Z = 160 X_1 + 240 X_2$
Α	$X_1 = 0, X_2 = 600$	$\therefore$ Z = Rs. 12, 000
	From Graph	
В	$X_1 = 150, X_2 = 150$	$\therefore$ Z = Rs. 10, 500
	Simultaneous Equations	
С	$X_1 = 300, X_2 = 0$	$\therefore$ Z = Rs. 15, 000
	From Graph	
D	$X_1 = 600, X_2 = 0$	$\therefore$ Z = Rs. 30, 000
	From Graph	

Min. Z = Rs. 10, 500

For B - B is at intersection of two constraint lines '6  $X_1$  + 2  $X_2 \ge 1200$ ' and ' $X_1$  +  $X_2$  = 300'.

 $\begin{array}{ll} 6 \ X_1 + 2 \ X_2 \geq 1200 & (1) \\ X_1 + X_2 = 300 & (2) \\ 2X_1 + 2X_2 = 600 & (2) \ X \ 2 \\ 2X_1 = 600 \\ X_1 = 150 \\ \text{Substituting value in Equation (2), } X_2 = 150. \\ \text{Solution} \\ \text{Optimal Cost} = \text{Rs. 10, 500/-} \\ X_1 = \text{No. of Units of P} = 150 \\ X_2 = \text{No. of Units of P} = 150. \end{array}$ 

### **EXERCISES**

1. What is meant by feasible region in graphical method.

2. What is meant by 'iso-profit' and 'iso-cost line' in graphical solution.

3. Mr. A. P. Ravi wants to invest Rs. 1, 00, 000 in two companies 'A' and 'B' so as not to exceed Rs. 75, 000 in either of the company. The company 'A' assures average return of 10% in whereas the average return for company 'B' is 20%. The risk factor rating of company 'A' is 4 on 0 to 10 scale whereas the risk factor rating for 'B' is 9 on similar scale. As Mr. Ravi wants to maximise his returns, he will not accept an average rate of return below 12% risk or a risk factor above 6.

Formulate this as LPP and solve it graphically.

4. Solve the following LPP graphically and interpret the result.

Max.  $Z = 8X_1 + 16 X_2$ Subject to:  $X_1 + X_2 \le 200$  $X_2 \le 125$  $3X_1 + 6X_2 \le 900$  $X_1, X_2 \ge 0$ 5. A furniture manufacturer makes two products - tables and chairs.

Processing of these products is done on two types of machines A and B. A chair requires 2 hours on machine type A and 6 hours on machine type B. A table requires 5 hours on machine type A and no time on Machine type B. There are 16 hours/day available on machine type A and 30 hours/day on machine type B. Profits gained by the manufacturer from a chair and a table are Rs. 2 and Rs. 10 respectively. What should be the daily production of each of the two products? Use graphical method of LPP to find the solution.

## LINEAR PROGRAMMING

Linear Programming - Special Properties, Interpretation Of Final Tableau, Concept Of Shadow Price, Concept Of Primal Dual Analysis

## Special Cases in Linear Programming

## a. Infeasible Solution (Infeasibility)

Infeasible means not possible. Infeasible solution happens when the constraints have contradictory nature. It is not possible to find a solution which can satisfy all constraints.

In graphical method, infeasibility happens when we cannot find Feasible region.



There is no common feasible region for line AB and CD. Hence, solution is infeasible.

## b. Unbounded Solution (Unboundedness)

Unbounded mean infinite solution. A solution which has infinity answer is called unbounded solution.

In graphical solution, the direction with respect to origin is as follows:



Now, in a maximisation problem, if we have following feasible region:



There is no upper limit (away from origin), hence the answer will be infinity. This is called unbounded solution.

#### c. Redundant Constraint (Redundancy)

A constraint is called redundant when it does not affect the solution. The feasible region does not depend on that constraint.

Even if we remove the constraint from the solution, the optimal answer is not affected.



The feasible region for the above problem is OABC. The 3rd constraint does not affect the feasible region.

Hence, the constraint  $X_1 \le 16$  is redundant constraint.

## d. Alternate Optimal Solution: (Multiple Optimal Solution)

Alternate or multiple optimal solution means a problem has more than one solution which gives the optimal answer.

There are two or more sets of solution values which give maximum profit or minimum cost. In graphical method, we come to know that there is optimal solution which is alternative when:

The iso-profit or iso-cost line is parallel to one of the boundaries of feasible region (they have the same slope value).

## SPECIAL CASES IN SIMPLEX

1. Unbounded Solution

 $\begin{array}{l} Max \ Z = 60 \ X_1 + 20 X_2 \Delta \\ \\ Subject \ to \\ 2 \ X_1 + 4 X_2 \geq 120 \\ 8 \ X_1 + 6 X_2 \geq 240 \\ \\ X_1, \ X_2 \geq 0 \\ \\ \hline \textbf{Solution} \\ \\ Max \ Z = 60 \ X_1 + 20 X_2 + 0 S_1 + 0 S_2 - M A_1 - M A_2 \\ \\ Subject \ to \\ 2 \ X_1 + 4 X_2 - S_1 + A_1 = 120 \\ 8 \ X_1 + 6 X_2 - S_2 + A_1 = 240 \\ \\ X_1, \ X_2, S_1, S_2, A_1, \ A_2, \geq 0 \end{array}$ 

When we solve this LPP by simplex method, we will get the following values in 4th Simplex Table.

Cj			60	20	0	0	- M	- M	R.R
С	Х	В	X1	$X_2$	<b>S</b> <sub>1</sub>	$S_2$	A <sub>1</sub>	A <sub>2</sub>	
0	<b>S</b> <sub>2</sub>	240	0	10	- 4	1			- 60
60	X1	60	1	2	-1/2	0			- 120
Z <sub>j</sub> —	→		60	120	- 30	0			
$\Delta = C_j$ -	Z <sub>j</sub> ——	►	0	- 100	30	0			
						•			

Max Positive  $C_j - Z_j = 30$ 

Key Column =  $S_1$ 

But there is no positive Replacement Ration R means there is an Entering variable, but there is no outgoing variable.

Hence, the solution is unbounded or infinity.

The value of Z (Profit) keeps on increasing infinitely.

#### 2. Infeasible Region

 $\begin{array}{l} Max \ Z = 3 \ X_1 + 2 \ X_2 \\ \\ Subject to: \\ X_1 + \ X_2 \leq 4 \\ 2 \ X_1 + \ X_2 \geq 10 \\ \\ X_1, \ X_2 \geq 0 \\ \\ Solution \\ \\ Standard \ Form \\ \\ Max \ Z = 3 \ X_1 + 2 \ X_2 + 0S_1 + 0S_2 - MA_1 \\ \\ Subject to \\ \\ X_1 + X_2 + S_1 = 4 \end{array}$ 

 $2 X_1 + X_2 - S_2 + A_1 = 10$ 

 $X_{1,} X_{2,} S_{1,} S_{2,} A_{1} \ge 0$ 

When we solve this LPP by simplex method, we will get the following values in 2nd Simplex Table.

Cj			60	20	0	0	- M	R.R
С	Х	В	X1	$X_2$	$\mathbf{S}_1$	$S_2$	$A_1$	< /
3	X1	4	1	1	1	0	0	$\bigtriangledown$
- M	A <sub>1</sub>	2	- 2	- 1	-1/2	- 1	1	
Z <sub>j</sub> —	→		3	3 + M	3 + 2M	М	- M	
$\Delta = C_j$ -	Z <sub>j</sub>	*	0	- 1 - M	-3 -2M	- M	0	

No positive  $\Delta$  value.

All  $C_j$  -  $Z_j$  values are either zero or negative. Hence, test of optimality is satisfied. So, the solution appears to be optimal. But an artificial variable (A<sub>1</sub>) is present in the basis, which has objective function coefficient of - M (infinity).

Hence, the solution is infeasible (Not feasible).

Infeasibility occurs when there is no solution which satisfies all the constraints of the LPP.

## **Concept of Shadow Price**

Shadow price of resource means value of one extra unit of resource. It is the maximum price the company should pay for procuring extra resources from market. It also indicates profitability or profit contribution of each resource (per unit).

Shadow price =  $'Z_i'$  value of slack variables.

 $S_1$  - slack variable of resource 1 and  $S_2$  - Slack Variable of resource 2. A slack variable represents unutilised capacity of a resource. Slack Variable is represented by 'S'.

## **Concept of Duality**

Every linear programming problem has a mirror image associated with it. If the original problem is maximisation, the mirror image is minimisation and vice versa.

The original problem is called 'primal' and the mirror image is called 'dual'.

The format of simplex method is such that when we obtain optimal solution of any one out of primal or dual, we automatically get optimal solution of the other.

For example, if we solve dual by simplex method, we also get optimal solution of primal.

## **Characteristics of Dual Problem**

1. Dual of the dual is primal.

2. If either the primal or dual has a solution, then the other also has a solution. The optimal value of both the solutions is equal.

3. If any of the primal or dual is infeasible then the other has an unbounded solution.

#### **Advantages of Duality**

1. If primal problem contains a large number of rows and a smaller number of columns we can reduce the computational procedure by converting into dual.

2. Solution of the dual helps in checking computational accuracy of the primal.

3. Economic interpretation of the dual helps the management in decision making. For example,

Minimisation LPP can be solved by two methods:

1. Simplex of Dual Method and

2. Artificial Variable Method

Method:1 Simplex of Dual Method

The original problem is called 'Primal'. We convert the problem in its 'Dual'.

Primal	Dual
1. Minimisation Problem Min. Z	Maximisation Problem Max. Z <sup>*</sup>
2. Constraints are of ' $\geq$ ' type	Constraints are of ' $\leq$ ' type.
3. Decision variables are $X_1$ , $X_2$ etc.	Decision variables are $Y_1$ , $Y_2$ etc.

4. Objective function coefficients of primal (4, 3) become RHS of constraints in Dual.

5. RHS of Constraints of Primal (4000, 50, 1400) become objective function coefficients of Dual.

6. In the LHS (left side) of constraints, all vertical values are written horizontally in Dual.

7. No. of Decision Variables in Primal = No. of constraints in Dual

8. No. of Constraints in Primal = No. of Decision Variables in Dual

For example,

Primal	Dual
Min Z = $4 X_1 + 3 X_2$	$Max Z = 4000 Y_1 + 50 Y_2 + 1400 Y_3$
Subject to:	Subject to:
$200 X_1 + 100 X_2 \ge 4000$	$200 y_1 + 1y_2 + 40y_3 \le 4$
$1 X_1 + 2 X_2 \ge 50$	$100 y_1 + 2y_2 + 40y_3 \le 3$
$40 X_1 + 40 X_2 \ge 1400$	

The numerical is calculated as shown in Simplex Method.

#### EXERCISES

1. Why an optimal solution to an unbounded maximisation LPP cannot be found in Simplex Method?

2. What is meant by shadow price of a resource?

#### SENSITIVITY ANALYSIS

The solution to a LPP is determined by simplex method is a static solution, It means that solution corresponds to:

- 1. Value of profit coefficients in the objective function, and
- 2. Availability of resources (i.e R.H.S of constraints)

But in reality, profit coefficients of variables may increase or decrease. Similarly, availability of resources may also increase or decrease. In that case, the optimal profit and optimal quantity (b values) of variables calculated as per Simplex solution will change.

The objective of sensitivity analysis is to determine the new values of solution. If possible, from the given simplex solution. This will be possible only if the changes (increase/decrease) in the objective function or constraint capacities is in certain limits. These will be two limits, lower limit (max. possible decrease) and upper limit (max. possible increase). These two limits provide the range within which the present simplex table remains optimal.

Hence, if the change in profit or change in capacity constraints is in the range, we can find new values of the solution. If it is not in the range, we cannot find the new values of the solution. Because the present simplex table will not remain optimal any more.

**EXAMPLE 1:** An electronics firm manufactures three products - transistors, resistors and capacitors which give profit of Rs. 100, 60 and 40 per unit respectively.

The firm uses three resources - Engineering, Direct Labour and Admin. capacities are 100, 600 and 300 hrs. respectively.

Cj			100	60	40	0	0	0
С	Х	В	$\mathbf{X}_1$	$X_2$	X3	$S_1$	$S_2$	<b>S</b> <sub>3</sub>
60	$X_2$	200/3	0	1	5/6	1	- 1/6	0
100	X1	100/3	1	0	1/6	0	1/6	0
0	<b>S</b> <sub>3</sub>	100	0	0	4		0	1
$Z_j \longrightarrow$			100	60	200/3	0	20/3	0
$\Delta = C_i - Z_i \longrightarrow$			0	0	-80/3	0	- 20/3	0

Following simplex solution is obtained.

Note: The solution is optimal as there is no positive C<sub>j</sub> - Z<sub>j</sub>.

## **Optimal Product Mix is:**

 $X_1 = 100/3 = No.$  of units of transistors.

 $X_2 = 200/3 = No.$  of units of resistors

 $X_3$  (capacitors) is not produced.

## **Optimal Profit is:**

Max. Z = [60 X 200/3] + [100 X 100/3] = [4000 + 10, 000/3] Max. Z = Rs. 22, 000/3

#### A. Range for profit coefficients of products:

Formula =  $\frac{\Delta j}{Xn}$ 

## **B.** Range for profit coefficients of X<sub>1</sub>:

Formula =  $\frac{\Delta j}{x_i}$ 

We take ration of ' $\Delta$ ' (C<sub>j</sub> - Z<sub>j</sub>) row and 'X<sub>1</sub>' row:

The ratios will be:

$$\frac{-80/3}{1/6} = \frac{-80}{3} \times \frac{6}{1} = -160$$
$$\frac{-\frac{100}{3}}{-2/3} = \frac{-100}{3} \times -\frac{3}{2} = 50$$
$$\frac{-20/3}{1/6} = \frac{-20}{3} \times \frac{6}{1} = -40$$

'-' sign indicates decrease in profit & '+' sign indicates increase in profit. It means possible decrease is 160 or 40. Hence, we can decrease profit by only 40. Possible increase is 50.  $\therefore$  Range for profit coefficient of X<sub>1</sub> is:

Original Profit  $\pm$  (increase or decrease)

100 + 50 = 150

100 - 40 = 60

: Rs. 60 to Rs. 150

It means profit of  $X_1$  (transistors) can fluctuate within the range of Rs. 60 to Rs. 150. The simplex solution will remain optimal in this range.

## 2. Range for profit coefficient of X<sub>2</sub>

Formula  $= \frac{\Delta 1}{X_2}$ We take ration of ' $\Delta$ ' (C<sub>j</sub> - Z<sub>j</sub>) row and 'X<sub>2</sub>' row:  $\frac{-80/3}{5/6} = \frac{-80}{3} \times \frac{6}{5} = -32$  $\frac{-\frac{100}{3}}{5/3} = \frac{-100}{3} \times -\frac{3}{5} = -20$  $\frac{-20/3}{1/6} = \frac{-20}{3} \times \frac{6}{1} = 40$ Hence, possible decrease is Rs. 20 and po

Hence, possible decrease is Rs. 20 and possible increase is Rs. 40. Range for profit coefficient of  $X_2$  is

60 - 20 = 40

60 + 40 = 100

Rs. 40 to Rs. 100.

It means the present simplex solution will remain optimal even if profit of  $X_2$  (resistors)fluctuates in the range of Rs. 40 to Rs. 100.

## 3. Range for profit coefficient of X<sub>3</sub>

X<sub>3</sub> capacitors is not produced.

 $\Delta \text{ of } X_3 = -80/3$ 

It means if we produce  $X_3$ , we will incur a loss of Rs. 80/3 per unit of  $X_3$ .So,  $X_3$  will be produced only if present profit of  $X_3$  is increased by Rs. 80/3.

 $\therefore$  X<sub>3</sub> will be produced if its profit becomes [40 + 80/3] = Rs. 200/3 or more than that. Hence, the present simplex solution remains optimal till the profit value of 200/3 for X<sub>3</sub>.  $\therefore$  Range for profit coefficient of X<sub>3</sub> is

Rs. zero to 200/3.

## **B.** Range for capacity of resources

OR Range for validity of shadow prices of Resources

Formula = -  $\left[\frac{b}{sn}\right]$ 

## 1. Range for capacity or availability of Engineering hours

Engineering hours is represented in simplex table by slack variable s1:

Formula = -  $\left[\frac{b}{S1}\right]$ 

We will take ratio of 'b' column and 'S<sub>1</sub>' column.

The ratios will be

$$-\left[\frac{200/3}{5/6}\right] = -\left[\frac{200}{3} \times \frac{3}{5}\right] = -40$$
$$-\left[\frac{100/3}{-2/3}\right] = -\left[\frac{100}{3} \times \frac{-3}{2}\right] = 50$$
$$-\left[\frac{100/3}{-2/3}\right] = -\left[-50\right] = 50$$

Hence, possible decrease in capacity is 40 hrs and possible increase in capacity is 50 hrs.

Range for resource capacity of Engineering is:

Original Capacity  $\pm$  (increase or decrease)

100 + 950) = 150100 - 40 = 60

 $\therefore$  Range is 60 hrs to 150 hrs.

It means the present simplex solution will remain optimal even if availability of Engineering resource fluctuates between 60 hrs to 150 hrs.

Note: In other words the shadow price of Engineering resource [which is Rs. 100/3] will remain valid even if the resource availability fluctuates between 60 hrs. to 150 hrs.

## 2. Range for availability of Direct Labour

Formula =  $\left[\frac{b}{52}\right]$ The ratios will be  $\left[\frac{200/3}{-1/6}\right] = -\left[\frac{200}{3}x\frac{-6}{1}\right] = 400$   $\left[\frac{100/3}{-1/6}\right] = -\left[\frac{100}{3}x\frac{6}{1}\right] = -200$   $\left[\frac{100}{0}\right] = \text{Infinity.}$   $\therefore$  Range for resource capacity of Direct Labour is: 600 + 400 = 1000

600 - 200 = 400

 $\therefore$  Range is 400 hrs to 1000 hrs.

It means the present simplex solution will remain optimal even if availability of Direct Labour resource fluctuates between 400 hrs to 1000 hrs.

Note: In other words, the shadow price of Direct Labour resource (which is Rs. 20/3) will remain valid even if the resource availability fluctuates between 400 hrs to 1000 hrs.

## 3. Range for availability of Admin:

Formula = - 
$$\left[\frac{b}{S3}\right]$$

The ratios will be

 $- \left[\frac{200/3}{0}\right] = - [Infinity] = Infinity$  $- \left[\frac{100/3}{0}\right] = - [Infinity] = Infinity$  $- \left[\frac{100}{1}\right] = - 100.$ 

It means the capacity can increase upto infinity.

Possible decrease = 100 hrs.

Range for resource capacity of Admin is

300 + Infinity = Infinity

300 - 100 = 200

 $\therefore$  Range is 200 hrs to Infinity.

It means the present simplex solution will remain optimal even if availability of Admin resource fluctuates between 200 hrs to Infinity.

Note: In other words, the shadow price of Admin resource (which is Rs. zero) will remain valid even if the resource availability fluctuates between 200 hrs to infinity.

c. Effect on the solution due to increase or decrease in the availability of resources

1. What will be the effect on solution if capacity of Engineering is increased by 30%? Answer:

Original capacity = 100 hrs.

Increase = 30%

New capacity = 130 hrs.

Note:

To find the effect on solution we need to find the range of resource capacity. We can find the effect on solution only if the new capacity is in the range.

If the new capacity goes out of the range, then we cannot find effect on solution.

Because in that case, the present simplex solution does not remain optimal any more.

From earlier calculation, we know that, Range for resource capacity of Engineering is 60 hrs to 150 hrs.

 $\therefore$  New capacity is in the range.

Change in capacity = 130 - 100 = +30 hrs.

Hence, we multiply column  $S_1$  by + 30.

From that we will get change in 'b' column.

	$S_1 X (+30) =$ Change in 'b' column
$X_2 \longrightarrow$	5/3 X + 30 = +50
$X_1 \longrightarrow$	-2/3 X + 30 = -20

Now we can find new basis values.

c	Х	New b
60	$X_2$	200/3 + 50 = 350/3
100	$X_1$	100/3 - 20 = 40/3

New Z =  $(60 \times 350/3) + (100 \times 40/3)$ 

New Z = 25, 000/3 Rs.

Increase in optimal profit = 25,000 - 22,000/3

= Rs. 1, 000

New Optimal Product Mix:

 $X_1 = 40/3$  units

 $X_2 = 350/3$  units.

2. Can you find out effect on optimal solution if excess capacity of abundant resource is transferred to Direct Labour?

Answer:

In the optional solution,  $S_3$  is present in the basis.

 $S_3 = 100$ 

 $S_3$  represents slack value of Admin.

Hence, Admin is abundant resource and its excess (unused) capacity is 100 hrs.

Capacity of Direct Labour = 600 hrs.

If 100 hrs are transferred,

New capacity of Direct labour = 700 hrs.

We, know that range of Direct Labour capacity is 400 hrs to 1000 hrs.

New capacity is in the range.

	$S_1 X (+30) =$ Change in 'b' column
$X_2 \longrightarrow$	-1/6 X + 100 = -100/650/3
$X_1 \longrightarrow$	1/6  X + 100 = 100/6 = 50/3

New Basis:

c	Х	New b
60	$X_2$	200/3 - 50/3 = 150/3 = 50
100	X1	100/3 + 50/3 = 150/3 = 50

New Z =  $(60 \times 50) + (100 + 50) = \text{Rs. } 8,000$ Increase in optimal profit = 8,000 - 22,000/3

= Rs. 2, 000/3

New Optimal Product Mix:

 $X_1 = 50$  units

 $X_2 = 50$  units.

3. What will be the effect on optimal solution if capacity of Admin is reduced to 175 hrs?

Answer: Range for capacity of Admin = 200 hrs to Infinity.

Since, 175 hrs is out of the range, if Admin capacity is reduced to 175 hr. solution will not remain optimal.

#### **EXERCISES**

1. An engineering company BMS Ltd. produces three products A, B and C using three machines  $M_1$ ,  $M_2$  and  $M_3$ . The resource constraints on  $M_1$ ,  $M_2$  and  $M_3$  are 96, 40 and 60 hours respectively. The profits earned by the products A, B and C are Rs. 2, Rs.5 and Rs. 8 per unit respectively. A simplex optimal solution to maximize the profit is given below where  $X_1$ ,  $X_2$  and  $X_3$  are quantities of products A, B and C produced by the company  $s_1$ ,  $s_2$  and  $s_3$  represent the slack in the resources  $M_1$ ,  $M_2$  and  $M_3$ . Study the solution given below and answer the following questions:

С	X Variables in the basis	X1	X <sub>2</sub>	X <sub>3</sub>	<b>S</b> <sub>1</sub>	<b>S</b> <sub>2</sub>	S <sub>3</sub>	B Solution Values
5	X2	1/3	1	0	1/6	- 1/3	0	8/3
8	X3	5/6	0	1	- 1/12	2/3	0	56/3
0	<b>S</b> <sub>3</sub>	7/3	0	0	- 1/13	- 1/3	1	44/3
	$\Delta = C - Z$	- 19/3	0	0	- 1/6	- 11/3	0	

1. Indicate the shadow price of each resource. Which of the resources are abundant and which are scarce?

2. What profit margin for product A do you expect the marketing department to secure if it is to be produced, and justify your advice?

3. Within what range, the profit of product B can change for the above solution to remain optimal?

4. How would an increase of 10 hours in the resource M<sub>2</sub> affect the optimality?

5. If the company BMS Ltd. wishes to raise production which of the three resources should be given priority for enhancement?

2. A business problem is formulated and expressed below as an LPP. (Profit is in Rs. and Resources are in units).

Objective function

Maximise  $Z = 80 X_1 + 100 X_2$ 

Subject to resource constraints,	
$X_1 + 2X_2 \le 720$	(Resource 1)
$5X_1 + 4X_2 \le 1800$	(Resource 2)
$3X_1 + X_2 \le 900$	(Resource 3)
$X_1 X_2 > 0$	

Simplex algorithm of LPP, applied to the above problem yielded following solution:

Basis							<b>B</b> <sub>1</sub>
C <sub>b</sub>	X <sub>b</sub>	$\mathbf{X}_1$	X2	<b>S</b> <sub>1</sub>	$S_2$	<b>S</b> <sub>3</sub>	
100	X2	0	1	5/6	- 1/6	0	300
80	X2	1	0	- 2/3	1/3	0	120
0	$S_3$	0	0	7/6	- 5/6	1	240
	Cj	80	100	0	0	0	
	$\Delta = C_j - Z_j$	0	0	- 30	- 10	0	

1. Answer the following questions with justification:

a. Is the solution optimal and unique?

b. Is the above solution infeasible?

•

c. What is the maximum profit as per optimal solution?

d. Which resources are abundant and which are scarce as per optimal solution?

2. Find out the range of coefficient of  $X_1$  in the objective function for which the above solution remains optimal.

3. Can you obtain the solution values of basic variables form the optimal solution when resource constraint (a) Changes to 750 units? If yes, find the new values of the basic variables.

# Linear programming : The Revised Simplex Method

## **The Revised Simplex Method**

While solving linear programming problem on a digital computer by regular simplex method, it requires storing the entire simplex table in the memory of the computer table, which may not be feasible for very large problem. But it is necessary to calculate each table during each iteration. The revised simplex method which is a modification of the original method is more economical on the computer, as it computes and stores only the relevant information needed currently for testing and / or improving the current solution. i.e. it needs only

- The net evaluation row  $\Delta_j$  to determine the non-basic variable that enters the basis.
- The pivot column
- The current basis variables and their values ( $X_B$  column) to determine the minimum positive ratio and then identify the basis variable to leave the basis.

The above information is directly obtained from the original equations by making use of the inverse of the current basis matrix at any iteration.

There are two standard forms for revised simplex method

- **Standard form-I** In this form, it is assumed that an identity matrix is obtained after introducing slack variables only.
- **Standard form-II** If artificial variables are needed for an identity matrix, then twophase method of ordinary simplex method is used in a slightly different way to handle artificial variables.

## Steps for solving Revised Simplex Method in Standard Form

```
Max Z = 2x_1 + x_2

Subject to

3 x_1 + 4 x_2 \le 6

6 x_1 + x_2 \le 3

and x_1, x_2 \ge 0

SLPP

Max Z = 2x_1 + x_2 + 0s_1 + 0s_2

Subject to

3 x_1 + 4 x_2 + s_1 = 6

6 x_1 + x_2 + s_2 = 3

and x_1, x_2, s_1, s_2 \ge 0
```

Solve by Revised simplex method

Step 1 - Express the given problem in standard form - I

- Ensure all  $b_i \ge 0$
- The objective function should be of maximization

• Use of non-negative slack variables to convert inequalities to equations The objective function is also treated as first constraint equation

and

Step 2 – Construct the starting table in the revised simplex form Express (1) in the matrix form with suitable notation

β₀ <sup>(1)</sup>			β1 <sup>(1)</sup>	β2 <sup>(1)</sup>		•	
el	a1 (1)	a <sub>2</sub> <sup>(1)</sup>	a <sub>3</sub> <sup>(1)</sup>	a4 <sup>(1)</sup>	<b>–</b> –		Xp
1	-2	-1	0	0 ]			
0	3	4	1	0	X2	=	6
0	6	1	0	1	s <sub>2</sub>		3

Column vector corresponding to Z is usually denoted by  $e_1$  matrix  $B_1$ , which is usually denoted as  $B_1 = [\beta_0^{(1)}, \beta_1^{(1)}, \beta_2^{(1)} \dots \beta_n^{(1)}]$ 

Hence the column  $\beta_0^{(1)}$ ,  $\beta_1^{(1)}$ ,  $\beta_2^{(1)}$  constitutes the basis matrix  $B_1$  (whose inverse  $B_1^{-1}$  is also  $B_1$ )

		$B_1^{-1}$				
Basic variables	e <sub>1</sub> (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X <sub>B</sub>	$X_k$	$X_B / X_k$
Z	1	0	0	0		
$\mathbf{s}_1$	0	1	0	6		
<b>S</b> <sub>2</sub>	0	0	1	3		

a <sub>1</sub> <sup>(1)</sup>	a <sub>2</sub> <sup>(1)</sup>
-2	-1
3	4
6	1

**Step 3** – Computation of  $\Delta_i$  for  $a_1^{(1)}$  and  $a_2^{(1)}$ 

 $\begin{array}{l} \Delta_1 = first \ row \ of \ B_1^{-1} \ast \ a_1^{(1)} = 1 \ \ast \ -2 + 0 \ \ast \ 3 + 0 \ \ast 6 = -2 \\ \Delta_2 = first \ row \ of \ B_1^{-1} \ast \ a_2^{(1)} = 1 \ \ast \ -1 + 0 \ \ast \ 4 + 0 \ \ast 1 = -1 \end{array}$ 

Step 4 – Apply the test of optimality

Both  $\Delta_1$  and  $\Delta_2$  are negative. So find the most negative value and determine the incoming vector.

Therefore most negative value is  $\Delta_1 = -2$ . This indicates  $a_1^{(1)}(x_1)$  is incoming vector.

**Step 5** – Compute the column vector  $X_k$ 

$$X_k = B_1^{-1} * a_1^{(1)}$$

ſ	1	0	0		-2		-2	
	0	1	0	*	3	=	3	
	_0	0	1_		6		_6_	

Step 6 – Determine the outgoing vector. We are not supposed to calculate for Z row.

	B1-1					
Basic variables	e <sub>1</sub> (Z)	β1 <sup>(1)</sup>	$\beta_2^{(1)}$	X <sub>B</sub>	X <sub>k</sub>	$X_B / X_k$
Z	1	0	0	0	-2	-
s <sub>1</sub>	0	1	0	6	3	2
s <sub>2</sub>	0	0	1	3	6	$1/2 \rightarrow outgoing$
					$\uparrow$	
					incoming	

Step 7 – Determination of improved solution

Column  $e_1$  will never change,  $x_1$  is incoming so place it outside the rectangular boundary

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	XB	$X_1$
$\mathbf{R}_1$	0	0	0	-2
$R_2$	1	0	6	3
R <sub>3</sub>	0	1	3	6

Make the pivot element as 1 and the respective column elements to zero.

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X <sub>B</sub>	X <sub>1</sub>
$R_1$	0	1/3	1	0
R <sub>2</sub>	1	-1/2	9/2	0
R <sub>3</sub>	0	1/6	1/2	1

Construct the table to start with second iteration

		$B_1^{-1}$				
Basic variables	e <sub>1</sub> (Z)	β1 <sup>(1)</sup>	$\beta_2^{(1)}$	X <sub>B</sub>	$X_k$	$X_B / X_k$
Z	1	0	1/3	1		
$s_1$	0	1	-1/2	9/2		
<b>x</b> <sub>1</sub>	0	0	1/6	1/2		

a4 <sup>(1)</sup>	a <sub>2</sub> <sup>(1)</sup>
0	-1
0	4
1	1

 $\Delta_4 = 1 * 0 + 0 * 0 + 1/3 * 1 = 1/3$  $\Delta_2 = 1 * -1 + 0 * 4 + 1/3 * 1 = -2/3$   $\Delta_2$  is most negative. Therefore  $a_2^{(1)}$  is incoming vector.

Compute the column vector

1	0 1/3		-1	2	-2/3
0	1 -1/2	*	4	Ш	7/2
0	0 1/6		1		1/6

Determine the outgoing vector

	B <sub>1</sub> <sup>-1</sup>					
Basic variables	e <sub>1</sub> (Z)	β1 <sup>(1)</sup>	$\beta_2^{(1)}$	X <sub>B</sub>	$X_k$	$X_B / X_k$
Z	1	0	1/3	1	-2/3	-
s <sub>1</sub>	0	1	-1/2	9/2	7/2	9/7→outgoing
x <sub>1</sub>	0	0	1/6	1/2	1/6	3
					<b>↑</b>	
					incoming	

Determination of improved solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X <sub>B</sub>	$X_2$
$R_1$	0	1/3	1	-2/3
$R_2$	1	-1/2	9/2	7/2
$R_3$	0	1/6	1/2	1/6

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	X <sub>B</sub>	$X_2$
$R_1$	4/21	5/21	13/7	0
$R_2$	2/7	-1/7	9/7	1
$R_3$	-1/21	8/42	2/7	0

		$B_1^{-1}$						
Basic	e <sub>1</sub>	<u>р</u> (1)	р (1)	V-	V.	$\mathbf{V}_{-} / \mathbf{V}_{-}$	$a_4^{(1)}$	$a_3^{(1)}$
variables	(Z)	$p_1$	$p_2$	ΛB	$\Lambda_k$	$\Lambda_{\rm B}/\Lambda_{\rm k}$		
Z	1	4/21	5/21	13/7			0	0
X2	0	2/7	-1/7	9/7			0	1
<b>X</b> 1	0	-1/21	8/42	2/7			1	0

 $\begin{array}{l} \Delta_4 = 1 \, * \, 0 + 4/21 \, * \, 0 + 5/21 \, * 1 = 5/21 \\ \Delta_3 = 1 \, * \, 0 + 4/21 \, * \, 1 + 5/21 \, * 0 = 4/21 \end{array}$ 

 $\Delta_4$  and  $\Delta_3$  are positive. Therefore optimal solution is Max Z = 13/7, x<sub>1</sub>= 2/7, x<sub>2</sub> = 9/7

## **11.3 Worked Examples**

#### **Example 1**

```
\begin{array}{l} Max \; Z = x_1 + 2x_2 \\ \text{Subject to} \\ & x_1 + x_2 \leq 3 \\ & x_1 + 2x_2 \leq 5 \\ & 3x_1 + x_2 \leq 6 \\ \text{and} \quad & x_1, x_2 \geq 0 \end{array}
```

#### Solution

#### SLPP

```
\begin{array}{l} Max \ Z = x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3\\ Subject \ to \\ x_1 + x_2 + s_1 = 3\\ x_1 + 2x_2 + s_2 = 5\\ 3x_1 + x_2 + s_3 = 6\\ and \quad x_1, x_2, s_1, s_2, s_3 \geq 0 \end{array}
```

#### **Standard Form-I**

 $\begin{array}{lll} Z-x_1-2x_2-0s_1-0s_2-0s_3=0\\ x_1+x_2+s_1+0s_2+0s_3=3\\ x_1+2x_2+0s_1+s_2+0s_3=5\\ 3x_1+x_2+0s_1+0s_2+s_3=6\\ \text{and} & x_1,x_2,s_1,s_2,s_3\geq 0 \end{array}$ 

#### Matrix form

β₀ <sup>(1)</sup>			β <sub>1</sub> <sup>(1)</sup>	β <sub>2</sub> <sup>(1)</sup>	β <sub>3</sub> <sup>(1)</sup>			
el	; a <sub>1</sub> (1)	$a_2^{(1)}$	: a <sub>3</sub> <sup>(1)</sup>	a4 <sup>(1)</sup>	$a_{5}^{(1)}$			
1	-1	-2	0	0	0		0	
0	1	1	1	0	0	X1 X2	3	
0	1	2	0	1	0	s <sub>1</sub> =	5	
_ 0	3	1	0	0	1	S2 S3	6	

## **Revised simplex table**

		В	-1 1				
Basic	e <sub>1</sub>	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X <sub>B</sub>	$X_k$	$X_B / X_k$
variables	(Z)						
Z	1	0	0	0	0		
s <sub>1</sub>	0	1	0	0	3		
s <sub>2</sub>	0	0	1	0	5		
\$3	0	0	0	1	6		

#### Additional table

a1 <sup>(1)</sup>	a <sub>2</sub> <sup>(1)</sup>
-1	-2
1	1
1	2
3	1

Computation of  $\Delta_j$  for  $a_1^{(1)}$  and  $a_2^{(1)}$ 

$$\Delta_1 = \text{first row of } B_1^{-1} * a_1^{(1)} = 1 * -1 + 0 * 1 + 0 * 1 + 0 * 3 = -1$$
  
$$\Delta_2 = \text{first row of } B_1^{-1} * a_2^{(1)} = 1 * -2 + 0 * 1 + 0 * 2 + 0 * 1 = -2$$

 $\Delta_2 = -2$  is most negative. So  $a_2^{(1)}(x_2)$  is incoming vector.

(	Compu	te the co	$\lim_{k \to B_1} ve$	$\cot X_1$	c I			
	1	0	0	<mark>0</mark> ک		<b>-</b> 2 ]	-2 ]	
	0	1	0	0		1	1	
	0	0	1	0	*	2	2	
	Lo	0	0	1		L <u>1</u> _	_1 _	

		В	-1 1				
Basic variables	e <sub>1</sub> (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	β <sub>3</sub> <sup>(1)</sup>	X <sub>B</sub>	X <sub>k</sub>	$X_B / X_k$
Z	1	0	0	0	0	-2	-
s <sub>1</sub>	0	1	0	0	3	1	3
s <sub>2</sub>	0	0	1	0	5	2	5/2→
<b>S</b> 3	0	0	0	1	6	$ \overline{1} $	6
						↑	

Improved Solution

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X <sub>B</sub>	X <sub>k</sub>
$\mathbf{R}_1$	0	0	0	0	-2
$R_2$	1	0	0	3	1
$R_3$	0	1	0	5	2
$R_4$	0	0	1	6	$\overline{1}$
184	U	U	I	U	1

	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_3^{(1)}$	X <sub>B</sub>	$X_k$
$R_1$	0	1	0	5	0
$R_2$	1	-1/2	0	1/2	0
$R_3$	0	1/2	0	5/2	1
$R_4$	0	-1/2	1	7/2	0

Revised simplex table for II iteration

		В	-1 1				
Basic variables	e <sub>1</sub> (Z)	$\beta_1^{(1)}$	$\beta_2^{(1)}$	$\beta_{3}^{(1)}$	X <sub>B</sub>	$X_k$	$X_B / X_k$
Ζ	1	0	1	0	5		
s <sub>1</sub>	0	1	-1/2	0	1/2		
<b>X</b> <sub>2</sub>	0	0	1/2	0	5/2		
S3	0	0 0 -1/2 1					

a <sub>1</sub> <sup>(1)</sup>	a4 <sup>(1)</sup>
-1	0
1	0
1	1
3	0

 $\begin{array}{l} \Delta_1 = 1 \, * \, \text{-}1 + 0 \, * \, 1 + 1 \, * 1 + 0 \, * 3 = 0 \\ \Delta_4 = 1 \, * \, 0 + 0 \, * \, 0 + 1 \, * 1 + 0 \, * 0 = 1 \end{array}$ 

 $\Delta_1$  and  $\Delta_4$  are positive. Therefore optimal solution is Max Z = 5,  $x_1{=}$  0,  $x_2{=}\,{5/2}$ 

# Linear programming : Computational Procedure of Dual Simplex Method

## **Introduction**

Any LPP for which it is possible to find infeasible but better than optimal initial basic solution can be solved by using dual simplex method. Such a situation can be recognized by first expressing the constraints in ' $\leq$ ' form and the objective function in the maximization form. After adding slack variables, if any right hand side element is negative and the optimality condition is satisfied then the problem can be solved by dual simplex method.

Negative element on the right hand side suggests that the corresponding slack variable is negative. This means that the problem starts with optimal but infeasible basic solution and we proceed towards its feasibility.

The dual simplex method is similar to the standard simplex method except that in the latter the starting initial basic solution is feasible but not optimum while in the former it is infeasible but optimum or better than optimum. The dual simplex method works towards feasibility while simplex method works towards optimality.

## **Computational Procedure of Dual Simplex Method**

The iterative procedure is as follows

Step 1 - First convert the minimization LPP into maximization form, if it is given in the minimization form.

Step 2 - Convert the ' $\geq$ ' type inequalities of given LPP, if any, into those of ' $\leq$ ' type by multiplying the corresponding constraints by -1.

Step 3 – Introduce slack variables in the constraints of the given problem and obtain an initial basic solution.

**Step 4** – Test the nature of  $\Delta_i$  in the starting table

- If all  $\Delta_j$  and  $X_B$  are non-negative, then an optimum basic feasible solution has been attained.
- If all  $\Delta_i$  are non-negative and at least one basic variable  $X_B$  is negative, then go to step 5.
- If at least  $\Delta_j$  one is negative, the method is not appropriate.

Step 5 – Select the most negative  $X_B$ . The corresponding basis vector then leaves the basis set B. Let  $X_r$  be the most negative basic variable.

Step 6 - Test the nature of  $X_r$ 

- If all X<sub>r</sub> are non-negative, then there does not exist any feasible solution to the given problem.
- If at least one  $X_r$  is negative, then compute Max  $(\Delta_j / X_r)$  and determine the least negative for incoming vector.

Step 7 – Test the new iterated dual simplex table for optimality.

Repeat the entire procedure until either an optimum feasible solution has been attained in a finite number of steps.

## 13.3 Worked Examples

#### Example 1

```
\begin{array}{l} \text{Minimize } Z = 2x_1 + x_2 \\ \text{Subject to} \\ 3x_1 + x_2 \geq 3 \\ 4x_1 + 3x_2 \geq 6 \\ x_1 + 2x_2 \geq 3 \\ \text{and } x_1 \geq 0, x_2 \geq 0 \end{array}
```

#### Solution

Step 1 – Rewrite the given problem in the form

 $\begin{array}{l} \text{Maximize } Z^{'} = & -2x_{1} - x_{2} \\ \text{Subject to} \\ & -3x_{1} - x_{2} \leq -3 \\ & -4x_{1} - 3x_{2} \leq -6 \\ & -x_{1} & -2x_{2} \leq -3 \\ & x_{1}, \, x_{2} \geq 0 \end{array}$ 

Step 2 – Adding slack variables to each constraint

 $\begin{array}{l} \text{Maximize } Z^{'} = & -2x_1 - x_2 \\ \text{Subject to} \\ & -3x_1 - x_2 + s_1 = -3 \\ & -4x_1 - 3x_2 + s_2 = -6 \\ & -x_1 - 2x_2 + s_3 = -3 \\ & x_1, \, x_2, \, s_1, s_2, \, s_3 \geq 0 \end{array}$ 

Step 3 – Construct the simplex table

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C <sub>B</sub>	X <sub>B</sub>	$X_1$	$X_2$	$\mathbf{S}_1$	$S_2$	$S_3$	
s <sub>1</sub>	0	-3	-3	-1	1	0	0	
s <sub>2</sub>	0	-6	-4	-3	0	1	0	$\rightarrow$ outgoing
S <sub>3</sub>	0	-3	-1	-2	0	0	1	
	Z' = 0		2	1	0	0	0	$\leftarrow \Delta_j$

**Step 4** – To find the leaving vector Min (-3, -6, -3) = -6. Hence s<sub>2</sub> is outgoing vector

Step 5 – To find the incoming vector

Max  $(\Delta_1 / x_{21}, \Delta_2 / x_{22}) = (2/-4, 1/-3) = -1/3$ . So x<sub>2</sub> is incoming vector

Step 6 – The key element is -3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C <sub>B</sub>	X <sub>B</sub>	X1	$X_2$	$\mathbf{S}_1$	$S_2$	$S_3$	
s <sub>1</sub>	0	-1	-5/3	0	1	-1/3	0	$\rightarrow$ outgoing
X2	-1	2	4/3	1	0	-1/3	0	
S3	0	1	5/3	0	0	-2/3	1	
			1					
	Z = -2		2/3	0	0	1/3	0	$\leftarrow \Delta_j$

Step 7 – To find the leaving vector

Min (-1, 2, 1) = -1. Hence s<sub>1</sub> is outgoing vector

Step 8 – To find the incoming vector

Max  $(\Delta_1 / x_{11}, \Delta_4 / x_{14}) = (-2/5, -1) = -2/5$ . So  $x_1$  is incoming vector

Step 9 – The key element is -5/3. Proceed to next iteration

	$C_j \rightarrow$		-2	-1	0	0	0	
Basic variables	C <sub>B</sub>	X <sub>B</sub>	X1	$X_2$	$\mathbf{S}_1$	<b>S</b> <sub>2</sub>	$S_3$	
X1	-2	3/5	1	0	-3/5	1/5	0	
X2	-1	6/5	0	1	4/5	-3/5	0	
<b>S</b> <sub>3</sub>	0	0	0	0	1	-1	1	
	Z' = -1	2/5	0	0	2/5	1/5	0	$\leftarrow \Delta_{i}$

Step 10 –  $\Delta_j \ge 0$  and  $X_B \ge 0$ , therefore the optimal solution is Max Z' = -12/5, Z = 12/5, and  $x_1=3/5, x_2=6/5$ 

#### Example 2

 $\begin{array}{l} \text{Minimize } Z = 3x_1 + x_2 \\ \text{Subject to} \\ x_1 + x_2 \geq 1 \\ 2x_1 + 3x_2 \geq 2 \\ \text{and} \ x_1 \geq 0, \, x_2 \geq 0 \end{array}$ 

#### Solution

 $\begin{array}{l} \text{Maximize } Z^{'} = & - 3 x_{1} - x_{2} \\ \text{Subject to} \\ & - x_{1} - x_{2} \leq -1 \\ & - 2 x_{1} - 3 x_{2} \leq -2 \\ & x_{1}, \, x_{2} \geq 0 \end{array}$ 

## SLPP

 $\begin{array}{l} \mbox{Maximize } Z^{'} = - \; 3x_1 - x_2 \\ \mbox{Subject to} \\ -x_1 - x_2 + s_1 = -1 \\ -2x_1 - \; 3x_2 + s_2 = -2 \\ x_1, \; x_2, \; s_1, s_2 \geq 0 \end{array}$ 

	$C_j \rightarrow$		-3	-1	0	0	
Basic variables	C <sub>B</sub>	X <sub>B</sub>	X <sub>1</sub>	X <sub>2</sub>	<b>S</b> <sub>1</sub>	<b>S</b> <sub>2</sub>	
<b>s</b> <sub>1</sub>	0	-1	-1	-1	1	0	
<b>s</b> <sub>2</sub>	0	-2	-2	-3	0	1	$\rightarrow$
				↑			
	Z' = 0		3	1	0	0	$\leftarrow \Delta_j$
S <sub>1</sub>	0	-1/3	-1/3	0	1	-1/3	$\rightarrow$
X2	-1	2/3	2/3	1	0	-1/3	l
						$\uparrow$	
	Z' = -2/3		7/3	0	0	1/3	$\leftarrow \Delta_j$
s <sub>2</sub>	0	1	1	0	-3	1	
X2	-1	1	1	1	-1	0	
	Z' = -1		2	0	1	0	$\leftarrow \Delta_j$

 $\Delta_j \ge 0$  and  $X_B \ge 0$ , therefore the optimal solution is Max  $Z^{'} = -1$ , Z = 1, and  $x_1 = 0$ ,  $x_2 = 1$ 

#### Notes on Sensitivity Analysis

Sensitivity analysis tells what would happen to the optimal solution if small changes were made in the problem. We consider the following types of changes.

- (1) Changes in **b**
- (2) Changes in c
- (3) Adding a variable
- (4) Deleting a variable
- (5) Deleting a constraint
- (6) Adding a constraint
- (7) Changes in **A**

I won't consider (1) and (2) here, because they are discussed in Chvátal.

I will use as an example the following linear programming problem:

maximize  $2x_1 + 2x_2 + x_3 - 3x_5$ subject to  $3x_1 + x_2 - x_5 \le 1$  $x_1 + x_2 + x_3 + x_4 \le 2$  $-3x_1 + 2x_3 + x_4 + 5x_5 \le 6$ 

all variables  $\geq 0$ 

The optimal dictionary is as follows:

 $x_3 = 1 + 2x_1 - x_4 - x_5 + s_1 - s_2$  $x_2 = 1 - 3x_1 + x_5 - s_1$ 

In terms of the Revised Simplex Method, the optimal basis is  $x_3$ ,  $x_2$ ,  $s_3$  with

$$B^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & -2 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \ \mathbf{y}^T = (1, 1, 0), \ \mathbf{t}_N^T = \begin{pmatrix} x_1 & x_4 & x_5 \\ 2 & 1 & 2 \end{pmatrix}.$$

Each of the examples below will start from this basis (i.e. the changes made in one section will not affect later sections).

#### (3) Adding a variable

This is easy using the Revised Simplex Method. We calculate the t value for the new variable. If that turns out to be negative, the original solution is not feasible for the dual problem (although of course it is still feasible for the primal). We then need a pivot with the new variable entering, as in the ordinary (Revised) Simplex Method. If you want to use dictionaries, use  $B^{-1}$  to calculate the coefficients of the new variable in the dictionary. After the first pivot, more pivots may be necessary until the basic solution is feasible for the dual problem.

In the example, let's add a new variable  $x_6$  which has coefficients 1, 2 and 3 in the constraints

and 4 in the objective, i.e.  $a_{16} = 1$ ,  $a_{26} = 2$ ,  $a_{36} = 3$ ,  $c_6 = 4$ . We get  $t_6 = (1, 1, 0) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 4 =$ 

-1 < 0, so  $x_6$  will enter the basis.  $\mathbf{d} = B^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ . Thus if you want to use dictionaries, you could insert  $x_6$  in the dictionary with the entries of  $-\mathbf{d}$  and  $-t_6$  as coefficients:

Or you could just continue with Revised Simplex. The ratios are 1 for  $x_3$ , 1 for  $x_2$  and 4 for  $s_3$ , so either  $x_3$  or  $x_2$  could leave. I'll choose  $x_3$  to leave. To update  $B^{-1}$  and  $\beta$ :

$$\begin{bmatrix} 1 & | & -1 & 1 & 0 & | & 1 \\ 1 & | & 1 & 0 & 0 & | & 1 \\ 1 & | & 2 & -2 & 1 & | & 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & | & -1 & 1 & 0 & | & 1 \\ 0 & | & 2 & -1 & 0 & | & 0 \\ 0 & | & 3 & -3 & 1 & | & 3 \end{bmatrix}$$
  
The new  $B^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 3 & -3 & 1 \end{pmatrix}$  and  $\beta = \begin{pmatrix} x_6 \\ x_2 \\ s_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ .  
 $\mathbf{y}^T = [4, 2, 0]B^{-1} = [0, 2, 0]$ .  $\mathbf{t}_N^T = [0, 2, 0]A_N - \mathbf{c}_N^T = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$   
Since all entries in  $\mathbf{y}^T$  and  $\mathbf{t}_N^T$  are  $\ge 0$ , this is an optimal solution.

#### (4) Deleting a variable

Deleting a variable means requiring it to be 0. In effect, it becomes an artificial variable. If the variable is already nonbasic, nothing needs to be done. If it is basic, however, it must leave the basis. This can be done with a "sign-reversed Dual Simplex pivot". It's "sign-reversed" because the variable starts off with a positive value, rather than a negative value which is usual in the Dual Simplex method. We can make it look like an ordinary Dual Simplex pivot if we multiply the equation by -1. After the first pivot, we can remove the deleted variable from the problem. More Dual Simplex pivots may be necessary until the basic solution is feasible for the primal problem.

In our example, suppose we want to delete the variable  $x_2$ , which is basic and has the value 1 in the optimal solution of the original problem. Multiply its equation by -1 to get  $-x_2 = -1 + 3x_1 - x_5 + s_1$ . Now do the Dual Simplex method with  $-x_2$  leaving. The ratios are 2/3 for  $x_1$  and 1 for  $s_1$ , so  $x_1$  enters. The next dictionary is

This is optimal (of course  $x_2$  is not allowed back in to the basis, in fact we can now remove it from the dictionary).

#### (5) Deleting a constraint

Deleting a constraint means that we no longer care about the value of its slack variable. In effect, that slack variable has become sign-free. If the slack variable was already basic, no pivoting is necessary, but if it is nonbasic (with a nonzero coefficient in the z equation) we will want it to enter the basis: increasing if the coefficient was positive (which would only happen if the constraint was an equality), or decreasing if the coefficient was negative. Recall (from the "Notes on the Simplex Method") that if a variable enters decreasing, we calculate ratios in the equations where the coefficient of the entering variable is positive. After the first pivot, we can delete the equation for the deleted constraint's slack variable from the dictionary. More pivots may be necessary until the basic solution is feasible for the primal problem.

In our example, suppose we delete the first constraint. Then  $s_1$ , which is nonbasic and has coefficient -1 in the z equation, enters decreasing. The only ratio to be calculated is for  $x_3$ , so  $x_3$  leaves the basis. The next dictionary is

We can delete the  $s_1$  equation. The solution is now optimal.

#### (6) Adding a constraint

The new constraint has a new slack variable. In the dictionary method, we write the equation for the new slack variable, but we need to substitute in the expressions for the basic variables so that we only have nonbasic variables on the right side.

In the Revised Simplex Method, if a new constraint is added we need a new row and column in the matrix  $B^{-1}$ . Making the new slack variable the last one in the basis, the new  $B^{-1}$  will look like this:

$$\begin{pmatrix} (\text{old } B^{-1}) & 0 \\ \mathbf{a}^T & 1 \end{pmatrix}$$

where if the coefficients of the basic variables in the new constraint form the row vector  $\alpha^T$ , then  $\mathbf{a}^T = -\alpha^T B^{-1}$ .

If the value of the new slack variable is negative (or if the new constraint is an equality and the value is nonzero), a Dual Simplex pivot will be needed.

In our example, we add the constraint  $x_1 + 2x_2 + 2x_3 \leq 3$ . The new slack variable is  $s_4 = 3 - x_1 - 2x_2 - 2x_3$ . Substituting the values for the basic variables  $x_2$  and  $x_3$  from the dictionary, we have  $s_4 = -1 + x_1 + 2x_4 + 2s_2$ . Thus the dictionary becomes

Using the Revised Simplex Method, the coefficients of the basic variables in the new constraint form the vector  $\alpha^T = [2, 2, 0]$ , so  $\mathbf{a}^T = -\alpha^T B^{-1} = [0, -2, 0]$ . Thus the new  $B^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix}$ 

 $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$  (with the old  $B^{-1}$  in the first three rows and columns, 0's to the right of it,

 $\mathbf{a}^{T}$  below and 1 in the bottom right). The new  $\beta = B^{-1} \begin{pmatrix} 1\\2\\6\\3 \end{pmatrix} = \begin{pmatrix} 1\\1\\4\\-1 \end{pmatrix}$ .

Because  $s_4 = -1$  in this dictionary, we need a Revised Simplex pivot with  $s_4$  leaving. The ratios are 2/1 for  $x_1$ , 1/2 for  $x_4$  and 1/2 for  $s_2$ . There is a tie for entering variable: I'll choose  $x_4$  to enter. The next dictionary is

This is optimal.

#### (7) Changing entries in A

There are two easy cases here, and one difficult case, depending on whether the constraints and variables where the entries are changed are basic or nonbasic.

(a) Suppose the changes are only in the coefficients of one or more nonbasic variables. Then it is as if the old versions of these variables were removed and new versions added, as in "Adding a variable" above. If the t entries for the new versions are negative, pivots will be necessary.

In our example, suppose we change the coefficient of  $x_5$  in the first constraint from -1 to -2. Think of this as introducing a new  $x_5$ . We calculate  $t_5 = (1,1,0) \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} - (-3) = 1 \ge 0$ , so the

optimal solution is unchanged.

(b) Suppose the changes are only in a constraint whose slack variable is basic. In the dictionary method, just as in "Adding a constraint" we use the dictionary equations to express the new version of this slack variable in terms of the nonbasic variables. If the value of this slack variable is negative, one or more Dual Simplex pivots will be needed.

In our example, suppose we change the coefficient of  $x_2$  in the third constraint from 0 to 2. The slack variable for the original version of the constraint had the dictionary equation  $s_3 = 4 - x_1 - 2x_4 - 3x_5 - 2s_1 + 2s_2$ . The new version is  $s'_3 = s_3 - 2x_2 = 2 + 5x_1 - 2x_4 - 5x_5 + 2s_2$ . This is still feasible, so no pivoting is necessary.

(c) Suppose the changes are for a basic variable and involve one or more constraints with nonbasic slack variables. Then the situation is more complicated. It is even possible that the *B* matrix may not be invertible (so that the current basis can no longer be used as a basis). Or if the basis can still be used, the solution may or may not be feasible for either the primal or the dual. One strategy that often works well is first to add a new version of the variable, and then delete the old version.

In our example, suppose we change the coefficient of  $x_2$  in the second constraint from 1 to 2. We first add the new version of  $x_2$ , call it  $x'_2$ , with coefficients 1, 2, 0 in the three constraints and

2 in the objective. Its t value is  $(1,1,0) \begin{pmatrix} 1\\ 2\\ 0 \end{pmatrix} - 2 = 1 \ge 0$ , so  $x'_2$  doesn't enter the basis. However,

we need to put it into the dictionary. We have  $\mathbf{d} = B^{-1} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ , so the dictionary is

Now we delete the old  $x_2$  in a "sign-reversed Dual Simplex" pivot. After changing signs in the  $x_2$  equation, we make  $-x_2$  leave. The ratios are 2/3 for  $x_1$ , 1/1 for  $x'_2$ , 1/1 for  $s_1$ , so  $x_1$  enters. The new dictionary is

$x_1$	=	1/3	$-1/3x_2$	$-1/3x'_{2}$		$+1/3x_{5}$	$-1/3s_1$	
$x_{i}$	3 =	5/3	$-2/3x_2$	$-5/3x'_{2}$	$-x_4$	$-1/3x_{5}$	$+1/3s_{1}$	$-s_2$
$s_3$	=	11/3	$+1/3x_{2}$	$+7/3x'_{2}$	$-2x_{4}$	$-10/3x_5$	$-5/3s_1$	$+2s_{2}$
z	=	7/3	$+2/3x_2$	$-1/3x_{2}'$	$-x_4$	$-8/3x_5$	$-1/3s_1$	$-s_2$

We delete the old  $x_2$ , and we have an optimal solution.

In some cases it would be useful to know if making the change will increase or decrease the objective value. Again, consider first adding the new version of the variable, and then deleting the old version. Note that primal simplex pivots increase the objective (or in the case of degeneracy may keep it the same, but never decrease it), while dual simplex pivots usually decrease the objective and never increase it.

- If the new version of the variable will not enter the basis when it's added, because its t value is not negative, then there is no primal simplex pivot but there will be at least one dual simplex pivot. So the change can't increase the objective, and will probably decrease it. This is what occurred in the example above, where the change decreased the objective from 3 to 7/3.
- If the new version of the variable does enter the basis because its t value is negative, the primal simplex pivots (if not degenerate) will increase the objective. If in this process the old version of the variable will leave the basis, there will be no need for dual simplex pivots to remove the old version, and so the objective can't decrease.

For example, suppose at the same time as we change the coefficient of  $x_2$  in the second constraint from 1 to 2 we also increase  $c_2$  (for the new version of the variable) from 2 to 4. Then  $t'_2$ decreases from 1 to -1, so the dictionary is

This time  $x'_2$  will enter the basis. Since this won't be a degenerate pivot, the objective will increase. There is a tie for minimum ratio between  $x_3$  and  $x_2$ ; we may as well choose  $x_2$  to leave. At this point we know that no dual simplex pivots will be needed to make  $x_2$  leave, so the change will increase the objective. In fact the next dictionary is

which is optimal. We can then delete the column for the old  $x_2$ . The objective has increased from 3 to 4.