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MANAGEMENT STUDIES

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DEPARTMENT OF BASIC SCIENCE
AND
HUMANITIES

Module -II

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER

11.1 Introduction

A differential equation of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right)=0$ in which the dependent variable $y(x)$ and its derivatives viz. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

11.2 Linear Differential Equations (LDE) with Constant Coefficients

A general linear differential equation of n^{th} order with constant coefficients is given by:

$$k_0 \frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + - - - - - + k_{n-1} \frac{dy}{dx} + k_n y = F(x)$$

where $k's$ are constant and $F(x)$ is a function of x alone or constant.

$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Or $f(D)y = F(x)$, where $D^n \equiv \frac{d^n}{dx^n}$, $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$, ….., $D \equiv \frac{d}{dx}$ are called differential operators.

11.3 Solving Linear Differential Equations with Constant Coefficients

Complete solution of equation $f(D)y = F(x)$ is given by $y =$ C.F + P.I.

where C.F. denotes complimentary function and P.I. is particular integral.

When $F(x) = 0$, then solution of equation $f(D)y = 0$ is given by $y =$ C.F

11.3.1 Rules for Finding Complimentary Function (C.F.)

Consider the equation $f(D)y = F(x)$

$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Step 1: Put $D = m$, auxiliary equation (A.E) is given by $f(m) = 0$

$\Rightarrow k_0 m^n + k_1 m^{n-1} + - - - - - + k_{n-1} m + k_n = 0$ ……③

Step 2: Solve the auxiliary equation given by ③

I. If the n roots of A.E. are real and distinct say $m_1, m_2, \dots m_n$
C.F. = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

II. If two or more roots are equal i.e. $m_1 = m_2 = \ldots = m_k,\ k \leq n$
C.F. = $(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_1 x} + \cdots + c_n e^{m_n x}$

III. If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i\beta,\ m_2 = \alpha - i\beta$

C.F. = $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

IV. If 2 pairs of imaginary roots are equal i.e. $m_1 = m_2 = \alpha + i\beta$,
$m_3 = m_4 = \alpha - i\beta$
C.F. = $e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + \cdots + c_n e^{m_n x}$

Example 1 Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3,5$

C.F. = $c_1 e^{3x} + c_2 e^{5x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$

Example 2 Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ …….①

By hit and trial $(m-2)$ is a factor of ①

∴① May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. = $c_1e^x + c_2e^{2x} + c_3e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x}$

Example 3 Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$ …….①

By hit and trial $(m - 1)$ is a factor of ①

∴① May be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m-1) - 9m^2(m-1) + 26m(m-1) - 24(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 9m^2 + 26m - 24) = 0$ ……②

By hit and trial $(m - 2)$ is a factor of ②

∴② May be rewritten as

$(m-1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m-1)[m^2(m-2) - 7m(m-2) + 12(m-2)] = 0$

$\Rightarrow (m-1)(m^2 - 7m + 12)(m-2) = 0$

$\Rightarrow (m-1)(m-3)(m-4)(m-2) = 0$

$\Rightarrow m = 1,2,3,4$

C.F. = $c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^x + c_2e^{2x} + c_3e^{3x} + c_4e^{4x}$

Example 4 Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m + 1)^2 = 0$

$\Rightarrow \ m = 0, -1, -1$

C.F. = $c_1 + \ (c_2 + c_3 x)e^{-x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y \ = \ \ c_1 + \ (c_2 + c_3 x)e^{-x}$

Example 5 Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + \ y = 0$

Solution: $\Rightarrow (D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m + 1)^2(m - 1)^2 = 0$

$\Rightarrow \ m = -1, -1, 1, 1$

C.F. = $(c_1 + c_2 x)e^{-x} + \ (c_3 + c_4 x)e^{x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y \ = \ \ (c_1 + c_2 x)e^{-x} + \ (c_3 + c_4 x)e^{x}$

Example 6 Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: $\Rightarrow (D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$ …….①

By hit and trial $(m + 2)$ is a factor of ①

∴① May be rewritten as

$m^3 + 2m^2 - 2m^2 - \ 4m + 2m + \ 4 = 0$

$\Rightarrow m^2(m + 2) - 2m(m + 2) + 2(m + 2) = 0$

$\Rightarrow (m + 2)(m^2 - 2m + 2) = 0$

$\Rightarrow \ m = -2, 1 \pm i$

C.F. = $c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{-2x} + e^x(c_2 \cos x + c_3 \sin x)$

Example 7 Solve the differential equation: $(D^2 - 2D + 5)^2 y = 0$

Solution: Auxiliary equation is: $(m^2 - 2m + 5)^2$.......①

Solving ①, we get

$\Rightarrow m = 1 \pm 2i, 1 \pm 2i$

C.F. = $e^x[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = e^x[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$

Example 8 Solve the differential equation: $(D^2 + 4)^3 y = 0$

Solution: Auxiliary equation is: $(m^2 + 4)^3$.......①

Solving ①, we get

$\Rightarrow m = \pm 2i, \pm 2i, \pm 2i$

C.F. = $(c_1 + c_2 x + c_3 x^2)\cos 2x + (c_4 + c_5 x + c_6 x^2)\sin 2x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2 x + c_3 x^2)\cos 2x + (c_4 + c_5 x + c_6 x^2)\sin 2x$

11.3.2 Shortcut Rules for Finding Particular Integral (P.I.)

Consider the equation $(D)y = F(x)$, $F(x) \neq 0$
$\Rightarrow (k_0 D^n + k_1 D^{n-1} + - - - - - + k_{n-1} D + k_n)y = F(x)$

Then P.I = $\frac{1}{f(D)} F(x)$, Clearly P.I. = 0 if $F(x) = 0$

Case I: When $F(x) = e^{ax}$

Use the rule P.I $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, $f(a) \neq 0$

In case of failure i.e. if $f(a) = 0$

P.I = $x\frac{1}{f'(D)}e^{ax} = x\frac{1}{f'(a)}e^{ax}$, $f'(a) \neq 0$

If $f'(a) = 0$, P.I. = $x^2\frac{1}{f''(a)}e^{ax}$, $f''(a) \neq 0$ and so on

Example 9 Solve the differential equation: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = e^{2x}$

Solution: $\Rightarrow (D^2 - 2D + 10)y = e^{2x}$

Auxiliary equation is: $m^2 - 2m + 10 = 0$

$\Rightarrow m = 1 \pm 3i$

C.F. = $e^x(c_1 \cos 3x + c_2 \sin 3x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)}e^{2x} = \frac{1}{f(2)}e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2)+10}e^{2x} = \frac{1}{10}e^{2x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = e^x(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10}e^{2x}$

Example 10 Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution: $\Rightarrow (D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2,1$

C.F. = $c_1e^{-2x} + c_2e^x$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)}e^x$, putting $D = 1, f(1) = 0$

$\therefore$P.I = $x\frac{1}{f'(D)}e^x$ $\because$ P.I. = $x\frac{1}{f'(a)}e^{ax}$ if $f(a) = 0$

$\Rightarrow$ P.I. = $x\frac{1}{2D+1}e^x = \frac{1}{f'(1)}e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. = $\frac{xe^x}{3}$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = c_1e^{-2x} + c_2e^{x} + \frac{xe^x}{3}$$

Example 11 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = \sinh(2x+1) + 4^x$

Solution: $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

C.F. = $c_1e^{2x} + c_2e^{-2x}$

P.I. = $\frac{1}{f(D)} F(x)$

$$= \frac{1}{f(D)}(\sinh(2x+1) + 4^x)$$

$$= \frac{1}{D^2-4}\left(\frac{e^{(2x+1)} - e^{-(2x+1)}}{2}\right) + \frac{1}{D^2-4}\left(e^{x\log 4}\right)$$

$$\because \sinh x = \frac{e^x - e^{-x}}{2} \text{ and } 4^x = e^{x\log 4}$$

$$= \frac{e}{2}\frac{1}{D^2-4}e^{2x} - \frac{e^{-1}}{2}\frac{1}{D^2-4}e^{-2x} + \frac{1}{D^2-4}e^{x\log 4}$$

Putting $D = 2, -2$ and $\log 4$ in the three terms respectively

$f(2) = 0$ and $f(-2) = 0$ for first two terms

$$\therefore \text{P.I.} = \frac{e}{2}x\frac{1}{2D}e^{2x} - \frac{e^{-1}}{2}x\frac{1}{2D}e^{-2x} + \frac{1}{(log 4)^2 - 4}e^{x\log 4}$$

$$\because \text{P.I.} = x\frac{1}{f'(a)}e^{ax} \text{ if } f(a) = 0$$

Now putting $D = 2, -2$ in first two terms respectively

$$\Rightarrow \text{P.I.} = \frac{ex}{8}e^{2x} + \frac{e^{-1}x}{8}e^{-2x} + \frac{4^x}{(log 4)^2 - 4} \qquad \because e^{x\log 4} = 4^x$$

$$\Rightarrow \text{P.I.} = \frac{x}{4}\left(\frac{e^{(2x+1)} + e^{-(2x+1)}}{2}\right) + \frac{4^x}{(log 4)^2 - 4}$$

$$\Rightarrow \text{P.I.} = \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log 4)^2 - 4} \qquad \because \cosh x = \frac{e^x + e^{-x}}{2}$$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2 - 4}$$

Case II: When $F(x) = Sin\,(ax+b) or\ Cos\,(ax+b)$

If $F(x) = Sin\,(ax+b)\ or\ Cos\,(ax+b)$, put $D^2 = -a^2$,

$D^3 = D^2D = -a^2D, D^4 = (D^2)^2 = a^4, \ldots\ldots$

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e. if $f(-a^2) = 0$

P.I. $= x\frac{1}{f'(-a^2)}\ Sin\,(ax+b)\ or\ Cos\,(ax+b), f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.$= x^2\frac{1}{f''(-a^2)}\ Sin\,(ax+b)\ or\ Cos\,(ax+b), f''(-a^2) \neq 0$

Example 12 Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow m = -2,1$

C.F. $= c_1e^{-2x} + c_2e^x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)}\sin x = \frac{1}{D^2+D-2}\sin x$

putting $D^2 = -1^2 = -1$

P.I $= \frac{1}{D-3}\sin x = \frac{D+3}{D^2-9}\sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2 = -1$

$\therefore$ P. I. $= \frac{-1}{10}(D\ sin\ x + 3\ \sin x)$

$= \frac{-1}{10}(\cos x + 3\ \sin x)$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1e^{-2x} + c_2e^{x} - \frac{1}{10}(\cos x + 3 \sin x)$

Example 13 Solve the differential equation: $(D^2 + 2D + 1)y = cos^2x$

Solution: Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$

$\Rightarrow m = -1, -1$

C.F. = $e^{-x}(c_1 + c_2x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)} cos^2x = \frac{1}{D^2+2D+1} \left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2} \frac{1}{D^2+2D+1} e^{0x} + \frac{1}{2} \frac{1}{D^2+2D+1} \cos 2x$

Putting $D = 0$ in the 1st term and $D^2 = -2^2 = -4$ in the 2nd term

P.I = $\frac{1}{2} + \frac{1}{2}\frac{1}{2D-3} \cos 2x$

$= \frac{1}{2} + \frac{1}{2} \frac{2D+3}{4D^2 - 3^2} \cos 2x$, Rationalizing the denominator

$= \frac{1}{2} + \frac{1}{2} \frac{(2D+3)\cos 2x}{-25}$, Putting $D^2 = -4$

$\therefore$ P. I. $= \frac{1}{2} - \frac{1}{50} (-4 \sin 2x + 3 \cos 2x)$

Now $y = $ C.F. + P.I

$\Rightarrow y = e^{-x}(c_1 + c_2x) + \frac{1}{2} - \frac{1}{50} (-4 \sin 2x + 3 \cos 2x)$

Example 14 Solve the differential equation: $(D^2 + 9)y = \sin 2x \cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

C.F. = $c_1 \cos 3x + c_2 \sin 3x$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{f(D)} \sin 2x \cos x = \frac{1}{2}\frac{1}{D^2+9} (\sin 3x + \sin x)$

$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \frac{1}{2}\frac{1}{D^2+9}\sin x$

Putting $D^2 = -9$ in the 1st term and $D^2 = -1$ in the 2nd term

We see that $f(D^2 = -9) = 0$ for the 1st term

$\therefore$ P.I. $= \frac{1}{2}\, x \frac{1}{2D}\sin 3x + \frac{1}{2}\frac{1}{8}\sin x$

$\because$ P.I. $= x\frac{1}{f'(-a^2)}\, Sin\,(ax+b)\,,\, f'(-a^2) \neq 0$

$\Rightarrow$P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Case III: When $F(x) = x^n$, n *is a positive integer*

P.I $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} x^n$

1. Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(\mathrm{D})]$ in the denominator and take it to numerator to become $[1 \pm \phi(\mathrm{D})]^{-1}$
2. Expand $[1 \pm \phi(\mathrm{D})]^{-1}$ using binomial theorem up to n^{th} degree as $(n+1)^{th}$ derivative of x^n is zero
3. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand $[1 \pm \phi(\mathrm{D})]^{-1}$in ascending powers of D

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Example 15 Solve the differential equation: $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution: $\Rightarrow (D^2 - 1)y = 5x - 2$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. = $c_1 e^{x} + c_2 e^{-x}$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-1}(5x-2)$

$= \frac{1}{-(1-D^2)}(5x-2)$

$= -(1-D^2)^{-1}(5x-2)$

$= -[1 + D^2 + \cdots](5x-2)$

$= -(5x-2)$

∴P.I = $-5x + 2$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ c_1 e^{x} + c_2 e^{-x} - 5x + 2$

Example 16 Solve the differential equation: $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is: $m^4 + 4m^2 = 0$

$\Rightarrow\ m^2(m^2+4) = 0$

$\Rightarrow m = 0, 0, \pm 2i$

C.F. = $(c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$

P.I. = $\frac{1}{f(D)}\ F(x)\ = \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{D^4+4D^2}(x^2+1)$

$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}(x^2+1)$

$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}(x^2+1)$

$= \frac{1}{4D^2}\left[1-\frac{D^2}{4}+\cdots\right](x^2+1)$

$= \frac{1}{4D^2}\left(x^2+1-\frac{1}{2}\right)$

$$= \frac{1}{4D^2}\left(x^2 + \frac{1}{2}\right)$$

$$= \frac{1}{4D}\int \left(x^2 + \frac{1}{2}\right) dx$$

$$= \frac{1}{4D}\left(\frac{x^3}{3} + \frac{x}{2}\right)$$

$$= \frac{1}{4}\int \left(\frac{x^3}{3} + \frac{x}{2}\right) dx$$

$$\therefore \text{P.I} = \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$$

Example 17 Solve the differential equation: $(D^2 - 6D + 9)y = 1 + x + x^2$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$$\Rightarrow (m - 3)^2 = 0$$

$$\Rightarrow m = 3,3$$

C.F. $= e^{3x}(c_1 + c_2 x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 6D + 9}(1 + x + x^2)$

$$= \frac{1}{9\left(1 - \frac{2D}{3} + \frac{D^2}{9}\right)}(1 + x + x^2)$$

$$= \frac{1}{9}\left(1 - \left(\frac{2D}{3} - \frac{D^2}{9}\right)\right)^{-1}(1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \left(\frac{2D}{3} - \frac{D^2}{9}\right) + \left(\frac{2D}{3} - \frac{D^2}{9}\right)^2 + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left(1 + x + x^2 + \frac{2}{3} + \frac{4x}{3} + \frac{2}{3}\right)$$

$\therefore$ P.I $= \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = e^{3x}(c_1 + c_2 x) + \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Case IV: When $F(x) = e^{ax} g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)} e^{ax} g(x) = e^{ax}\left(\frac{1}{f(D+a)} g(x)\right)$

Example 18 Solve the differential equation: $(D^2 + 2)y = x^2 e^{3x}$

Solution: Auxiliary equation is: $m^2 + 2 = 0$

$\Rightarrow m^2 = -2$

$\Rightarrow m = \pm\sqrt{2} i$

C.F. = $(c_1 \cos(\sqrt{2}\, x) + c_2 \sin(\sqrt{2}\, x))$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+2} x^2 e^{3x}$

$= e^{3x} \frac{1}{(D+3)^2+2} x^2$

$= e^{3x} \frac{1}{D^2+6D+11} x^2$

$= \frac{e^{3x}}{11} \frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)} x^2$

$= \frac{e^{3x}}{11}\left(1 + \left(\frac{6D}{11} + \frac{D^2}{11}\right)\right)^{-1} x^2$

$= \frac{e^{3x}}{11}\left[1 - \left(\frac{6D}{11} + \frac{D^2}{11}\right) + \left(\frac{6D}{11} + \frac{D^2}{11}\right)^2 + \cdots\right] x^2$

$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121} + \cdots\right] x^2$

$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} + \frac{25D^2}{121} + \cdots\right] x^2$

$= \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

$\therefore\ P.I\ = \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Complete solution is: $y\ =\ $ C.F. + P.I

$\Rightarrow y\ =\ (c_1\cos(\sqrt{2}\,x) + c_2\sin(\sqrt{2}\,x)) + \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Example 19 Solve the differential equation: $(D^3+1)y = e^{2x}\sin x$

Solution: Auxiliary equation is: $m^3 + 1 = 0$

$\Rightarrow m^3 = -1$

$\Rightarrow m = -1, \frac{1\pm\sqrt{3}i}{2}$

C.F. = $c_1 e^{-x} +\ e^{\frac{x}{2}}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^3+1}e^{2x}\sin x$

$= e^{2x}\frac{1}{(D+2)^3+1}\sin x$

$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x$

$= e^{2x}\frac{1}{-D-6+12D+9}\sin x$, Putting $D^2 = -1$

$= e^{2x}\frac{1}{11D+3}\sin x$

$= e^{2x}\frac{11D-3}{121D^2-9}\sin x$, Rationalizing the denominator

$= -\frac{e^{2x}}{130}(11D-3)\sin x$, Putting $D^2 = -1$

$\therefore$ P.I $\ = -\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Complete solution is: $y\ =\ $ C.F. + P.I

$\Rightarrow y\ = c_1 e^{-x} +\ e^{\frac{x}{2}}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$-\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Example 20 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = x\ \sinh x$

Solution: $\Rightarrow (D^2 - 4)y = x\ \sinh x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow\ m = \pm 2$

C.F. = $c_1 e^{2x} + c_2 e^{-2x}$

P.I. = $\frac{1}{f(D)}\ F(x)$

$$= \frac{1}{f(D)}(x\ \sinh x)$$

$$= \frac{1}{D^2-4}\left(x\,\frac{e^x - e^{-x}}{2}\right) \qquad \because \sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{D^2-4}\left(x\,\frac{e^x}{2} - x\,\frac{e^{-x}}{2}\right)$$

$$= \frac{e^x}{2}\,\frac{1}{(D+1)^2-4}\,x - \frac{e^{-x}}{2}\,\frac{1}{(D-1)^2-4}\,x$$

$$= \frac{e^x}{2}\,\frac{1}{(D^2+2D-3)}\,x - \frac{e^{-x}}{2}\,\frac{1}{D^2-2D-3}\,x$$

$$= \frac{e^x}{2}\,\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)}\,x - \frac{e^{-x}}{2}\,\frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}\,x$$

$$= -\frac{e^x}{6}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1} x + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1} x$$

$$= -\frac{e^x}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$$

$$= -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right)$$

$\therefore$ P.I. $= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Example 21 Solve the differential equation: $(D^2+1)y = x^2 \sin 2x$

Solution: Auxiliary equation is: $m^2+1=0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

C.F. = $c_1 \cos x + c_2 \sin x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} x^2 \sin 2x$

= Imaginary part of $\frac{1}{D^2+1} x^2 e^{i2x}$

Now $\frac{1}{D^2+1} x^2 e^{i2x} = e^{i2x} \frac{1}{(D+2i)^2+1} x^2$

$$= e^{i2x} \frac{1}{D^2+4i^2+4iD+1} x^2$$

$$= e^{i2x} \frac{1}{D^2+4iD-3} x^2$$

$$= e^{i2x} \frac{1}{-3\left(1-\frac{D^2}{3}-\frac{4iD}{3}\right)} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\left(\frac{D^2}{3}+\frac{4iD}{3}\right)\right]^{-1} x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)^2\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\frac{D^2}{3}+\frac{4iD}{3}+\frac{16i^2D^2}{9}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\frac{13D^2}{9}+\frac{4iD}{3}\right] x^2$$

$$= \frac{-e^{i2x}}{3}\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$$= -\frac{1}{3}(\cos 2x + i \sin 2x)\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$\therefore$ P.I. = Imaginary part of $\frac{1}{D^2+1} x^2 e^{i2x} = -\frac{1}{3}\left(\frac{8x}{3}\cos 2x + \left(x^2-\frac{26}{9}\right)\sin 2x\right)$

$$= -\frac{8x}{9}\cos 2x + \frac{1}{27}(26-9x^2)\sin 2x$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{8x}{9} \cos 2x + \frac{1}{27}(26 - 9x^2) \sin 2x$$

Example 22 Solve the differential equation: $(D^2 - 4D + 4)y = x^2 e^{2x} \sin 2x$

Solution: Auxiliary equation is: $m^2 - 4m + 4 = 0$

$\Rightarrow (m - 2)^2$

$\Rightarrow m = 2,2$

C.F. = $(c_1 + c_2 x)e^{2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2-4D+4} x^2 e^{2x} \sin 2x$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= e^{2x} \frac{1}{D} \left[(x^2)\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right) \right]$$

$$= e^{2x} \frac{1}{D} \left[-\frac{1}{2} x^2 \cos 2x + \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x \right]$$

$$= e^{2x} \left[-\frac{1}{2} \int x^2 \cos 2x + \frac{1}{2} \int x \sin 2x \, dx + \frac{1}{4} \int \cos 2x \, dx \right]$$

$$= e^{2x} \left[-\frac{1}{2} \left[(x^2)\left(\frac{\sin 2x}{2}\right) - (2x)\left(\frac{-\cos 2x}{4}\right) + (2)\left(\frac{-\sin 2x}{8}\right) \right] + \right.$$

$12x - \cos 2x2 - 1 - \sin 2x4 + 14 \sin 2x2$

$$\therefore \text{P.I.} = e^{2x} \left[\frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x \right]$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = (c_1 + c_2 x)e^{2x} + e^{2x} \left[\frac{-x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x \right]$$

Case V: When $F(x) = x\, g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}\big(x\, g(x)\big) = x \frac{1}{f(D)} g(x) + \left(\frac{d}{dD} \frac{1}{f(D)} \right) g(x)$

Example 23 Solve the differential equation: $(D^2+9)y = x\cos x$

Solution: Auxiliary equation is: $m^2+9=0$

$\Rightarrow m^2 = -9$

$\Rightarrow m = \pm 3i$

C.F. = $(c_1\cos 3x + c_2 \sin 3x)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2+9}\ x\cos x$

$= x\frac{1}{D^2+9}\ \cos x + \frac{-2D}{(D^2+9)^2}\ \cos x$

$= x\frac{1}{-1+9}\ \cos x + \frac{-2D}{(-1+9)^2}\ \cos x,$ Putting $D^2 = -1$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$= \frac{x\cos x}{8} - \frac{2D\cos x}{64}$

$\therefore$ P.I. $= \frac{x\cos x}{8} + \frac{\sin x}{32}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y\ = c_1\cos 3x + c_2\sin 3x + \frac{x\cos x}{8} + \frac{\sin x}{32}$

Example 24 Solve the differential equation:

$$(D^2-1)y = x\sin x + (1+x^2)e^x$$

Solution: Auxiliary equation is: $m^2-1=0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^x + c_2e^x$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-1}\ (x\sin x + (1+x^2)e^x)$

$= \frac{1}{D^2-1}x\sin x + \frac{1}{D^2-1}(1+x^2)e^x$

Now $\frac{1}{D^2-1}x\sin x = = x\frac{1}{D^2-1}\ \sin x + \frac{-2D}{(D^2-1)^2}\ \sin x$

$$= x\frac{1}{-1-1}\sin x + \frac{-2D}{(-1-1)^2}\sin x, \qquad \text{Putting } D^2 = -1$$

$$= -\frac{1}{2}(x\sin x + \cos x)$$

Also $\frac{1}{D^2-1}(1+x^2)e^x = e^x\frac{1}{(D+1)^2-1}(1+x^2)$

$$= e^x\frac{1}{D^2+2D}(1+x^2)$$

$$= e^x\frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= e^x\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= e^x\frac{1}{2D}\left[1-\frac{D}{2}+\frac{D^2}{4}\right](1+x^2)$$

$$= e^x\frac{1}{2D}\left[1+x^2-x+\frac{1}{2}\right]$$

$$= e^x\frac{1}{2D}\left[x^2-x+\frac{3}{2}\right]$$

$$= \frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$$

$\therefore$ P.I. $= -\frac{1}{2}(x\sin x+\cos x)+\frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = c_1e^x + c_2e^x - \frac{1}{2}(x\sin x+\cos x)+\frac{e^x}{2}\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}\right]$$

Case VI: When $F(x)$ is any general function of x not covered in shortcut methods I to V above

Resolve $f(D)$ into partial fractions and use the rule:

$$\frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$$

Example 25 Solve the differential equation: $(D^2+3D+2)y = e^{e^x}$

Solution: Auxiliary equation is: $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0$

$\Rightarrow m = -1, -2$

C.F. = $c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+1)(D+2)} e^{e^x}$

$= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x}$

$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx$

$= e^{-x} \int De^{e^x} dx - e^{-2x} \int e^x De^{e^x} dx$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int e^x e^{e^x} dx\right]$, Integrating 2nd term by parts

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int De^{e^x} dx\right]$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - e^{e^x}\right]$

$\therefore$ P.I. $= e^{-2x} e^{e^x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

Example 26 Solve the differential equation: $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x$

$= \frac{1}{(D-2i)(D+2i)} \tan 2x$

$= \frac{1}{4i}\left(\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right) \tan 2x$

P.I. $= \frac{1}{4i}\left(\frac{1}{D-2i} \tan 2x\right) - \frac{1}{4i}\left(\frac{1}{D+2i} \tan 2x\right)$ ....①

Now $\frac{1}{D-2i}\tan 2x = e^{2ix}\int e^{-2ix}\tan 2x\,dx$

$$= e^{2ix}\int(\cos 2x - i\sin 2x)\tan 2x\,dx$$

$$= e^{2ix}\int(\sin 2x - i\frac{sin^2 2x}{cos2x})\,dx$$

$$= e^{2ix}\int\left(\sin 2x - i\frac{1-cos^2 2x}{cos2x}\right)dx$$

$$= e^{2ix}\int(\sin 2x - i\sec 2x + i\cos 2x\,)\,dx$$

$$= e^{2ix}\left(-\frac{1}{2}\cos 2x - \frac{i}{2}log|\sec 2x + \tan 2x| + \frac{i}{2}\sin 2x\right)$$

$$\therefore \frac{1}{D-2i}\tan 2x = e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right) \ldots②$$

Replacing i by $-i$

$$\frac{1}{D+2i}\tan 2x = e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right) \ldots③$$

Using ②and ③ in ①

$$\text{P.I.} = \frac{1}{4i}\left[e^{2ix}\left(-\frac{1}{2}e^{-2ix} - \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$
$$-\frac{1}{4i}\left[e^{-2ix}\left(-\frac{1}{2}e^{2ix} + \frac{i}{2}log|\sec 2x + \tan 2x|\right)\right]$$

$$= \frac{1}{4i}\left[-\frac{1}{2} - \frac{i}{2}e^{2ix}log|\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2}e^{-2ix}log|\sec 2x + \tan 2x|\right]$$

$$= \frac{1}{4i}\left[-i\frac{e^{2ix} + e^{-2ix}}{2}log|\sec 2x + \tan 2x|\right]$$

$$\therefore \text{P.I} = -\frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Complete solution is: y = C.F. + P.I.

$$\Rightarrow y = c_1\cos 2x + c_2\sin 2x - \frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$$

Exercise 11A

Solve the following differential equations:

1. $(D^3 + D^2 - 5D + 3)y = 0$ Ans. $y = (c_1x + c_2)e^x + c_3e^{-3x}$

2. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{3x}$ Ans. $y = c_1e^{2x} + c_2e^{3x} + e^{3x}(x-1)$

3. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{x}\cosh 2x$

Ans.$y = c_1e^{-3x} + c_2e^{2x} + \frac{1}{12}e^{3x} - \frac{1}{12}e^{-x}$

4. $(D-1)^2(D^2+1)^2y = e^x$

Ans. $y = (c_1x + c_2)e^x + (c_3x + c_4)\cos x + (c_5x + c_6)\sin x + \frac{x^2}{8}e^x$

5. $(D^2 - 6D + 9)y = x^2 + 2e^{2x}$

Ans. $y = (c_1x + c_2)e^{3x} + \frac{1}{9}\left(x^2 + \frac{4x}{8} + \frac{2}{3}\right) + 2e^{2x}$

6. $(D^2 + D - 2)y = x + \sin x$

Ans. $y = c_1e^{-2x} + c_2e^{x} - \frac{1}{4}(2x+1) - \frac{1}{10}(\cos x + 3\sin x)$

7. $(D^2 + D)y = (1 + e^x)^{-1}$

Ans.$y = c_1 + c_2e^{-x} + x - (1 + e^{-x})\log(1 + e^x)$

8. $(D^2 + 5D + 6)y = e^{-2x}sec^2x\,(1{+}2\tan x)$

Ans. $y = c_1e^{-2x} + c_2e^{-3x} + e^{-2x}(\tan x - 1)$

9. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x-1)e^{2x}$

Ans. $y = c_1e^{2x} + c_2e^{-6x} + \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9x}{8}\right)$

10. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x + e^{3x}$, given $y = 1, \frac{dy}{dx} = -1$ when $x = 0$

Ans. $y = -\frac{1}{2}e^{x} - 2e^{2x} + 2x + 3 + \frac{e^{3x}}{2}$

11.4 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

11.4.1 Cauchy's Linear Differential Equation

The differential equation of the form:

$$k_0\,x^n\frac{d^ny}{dx^n} + k_1\,x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + - - - - + k_{n-1}\,x\frac{dy}{dx} + k_n\,y = F(x)$$

is called Cauchy's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$x = e^t \quad \Rightarrow \log x = t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} = Dy$, where $D \equiv \frac{d}{dt}$

Similarly $x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Example 27 Solve the differential equation:

$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\cos(\log x), x > 0$ ....①

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$ and $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

∴① May be rewritten as

$(D(D-1)(D-2) + 3D(D-1) + D + 8)y = 13\cos t$

$\Rightarrow (D^3 + 8)y = 13\cos t \ , D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

C.F. = $c_1e^{-2t} + e^t\left(c_2\cos\sqrt{3}t + c_3\sin\sqrt{3}t\right)$

$= \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)\right)$

P.I. $= \frac{1}{f(D)}F(x) = 13\frac{1}{D^3+8}\cos t$

$= 13\frac{1}{-D+8}\cos t$, Putting $D^2 = -1$

$= 13\frac{(8+D)}{64-D^2}\cos t = 13\frac{(8+D)}{65}\cos t$ Putting $D^2 = -1$

$\therefore$ P. I. $= \frac{1}{5}\,(8\cos t\ + D\cos t)$

$= \frac{1}{5}\,(8\cos t - \sin t)$

$= \frac{1}{5}\,(8\cos(\log x) - \sin(\log x))$

Complete solution is: $y\ =\ $ C.F. + P.I

$$\Rightarrow y\ =\ \frac{c_1}{x^2} + x\big(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x\big) +$$

$$\frac{1}{5}\,(8\cos(\log x) - \sin(\log x))$$

Example 28 Solve the differential equation:

$$x^2\frac{\mathrm{d^2y}}{\mathrm{dx^2}} + x\frac{\mathrm{dy}}{\mathrm{dx}} - y\ = \frac{x^3}{1+x^2} \qquad \ldots\ldots.①$$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{\mathrm{d^2y}}{\mathrm{dx^2}} = \mathrm{D(D-1)y}$

$\therefore$① May be rewritten as

$(\mathrm{D(D-1) + D - 1})y = \frac{e^{3t}}{1+e^{2t}}$

$\Rightarrow (D^2 - 1)y = \frac{e^{3t}}{1+e^{2t}}\,,\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. $=\ c_1e^{-t} + c_2e^t$

$=\ \frac{c_1}{x} + c_2x$

P.I. $= \frac{1}{f(\,D)}\ F(x) =\ \frac{1}{D^2-1}\ \frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{(D-1)(D+1)}\ \frac{e^{3t}}{1+e^{2t}} = \frac{1}{2}\left(\frac{1}{(D-1)} - \ \frac{1}{(D+1)}\right)\ \frac{e^{3t}}{1+e^{2t}}$

$= \frac{1}{2}\left(\frac{1}{(D-1)}\frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)}\frac{e^{3t}}{1+e^{2t}}\right)$

$= \frac{1}{2}\left(e^{t}\int e^{-t}\frac{e^{3t}}{1+e^{2t}}dt - e^{-t}\int e^{t}\frac{e^{3t}}{1+e^{2t}}dt\right) \because \frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$

$= \frac{1}{2}\left(e^{t}\int \frac{e^{2t}}{1+e^{2t}}dt - e^{-t}\int \frac{e^{4t}}{1+e^{2t}}dt\right)$

Put $e^{2t} = u \Rightarrow 2e^{2t}dt = du$

$\therefore \text{P.I} = \frac{1}{4}\left(e^{t}\int \frac{1}{1+u}du - e^{-t}\int \frac{u}{1+u}du\right)$

$= \frac{1}{4}\left(e^{t}\log(1+u) - e^{-t}\int \frac{1+u-1}{1+u}du\right)$

$= \frac{1}{4}\left(e^{t}\log(1+u) - e^{-t}\int \left(1-\frac{1}{1+u}\right)du\right)$

$= \frac{1}{4}(e^{t}\log(1+u) - e^{-t}(u-\log(1+u))$

$= \frac{1}{4}(e^{t}\log(1+e^{2t}) - e^{-t}(e^{2t}-\log(1+e^{2t}))$

$= \frac{1}{4}\left(x\log(1+x^2) - \frac{1}{x}(x^2-\log(1+x^2))\right)$

$= \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = \frac{c_1}{x} + c_2 x + \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2) - \frac{x}{4}$

$\Rightarrow y = \frac{c_1}{x} + c_3 x + \frac{1}{4}\left(x+\frac{1}{x}\right)\log(1+x^2)$, $c_3 = c_2 - \frac{1}{4}$

Example 29 Solve the differential equation:

$x^2D^2 - 2x\text{D} - 4y = x^2 + 2\log x, \quad x > 0$ ....①

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow xD = \delta y, x^2D^2 = \delta(\delta-1)\text{y}$, $\delta \equiv \frac{d}{dt}$

∴① May be rewritten as

$(\delta(\delta-1)-2\delta-4)y = e^{2t}+2t$

$\Rightarrow (\delta^2-3\delta-4)y = e^{2t}+2t$

Auxiliary equation is: $m^2-3m-4=0$

$\Rightarrow (m+1)(m-4)=0$

$\Rightarrow\ m=-1,4$

C.F. = $c_1e^{-t}+c_2e^{4t}$

$=\frac{c_1}{x}+\frac{c_2}{x^4}$

P.I. = $\frac{1}{f(\delta)}\,F(x) = \frac{1}{\delta^2-3\delta-4}(e^{2t}+2t)$

$=\frac{1}{\delta^2-3\delta-4}e^{2t}+\frac{1}{\delta^2-3\delta-4}2t$

$=\frac{1}{-6}e^{2t}+2\frac{1}{-4\left(1-\frac{\delta^2}{4}+\frac{3\delta}{4}\right)}t$ Putting $\delta=2$ in the 1st term

$=\frac{-e^{2t}}{6}-\frac{1}{2}\left(1-\left(\frac{\delta^2}{4}-\frac{3\delta}{4}\right)\right)^{-1}t$

$=\frac{-e^{2t}}{6}-\frac{1}{2}\left[1+\frac{\delta^2}{4}-\frac{3\delta}{4}+\cdots\right]t$

$=\frac{-e^{2t}}{6}-\frac{1}{2}\left[t-\frac{3}{4}\right]$

$\therefore$ P.I. $=\frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = \frac{c_1}{x}+\frac{c_2}{x^4}\ \frac{-x^2}{6}-\frac{1}{2}\left[\log x-\frac{3}{4}\right]$

11.4.2 Legendre's Linear Differential Equation

The differential equation of the form: $k_0(ax+b)^n\frac{\mathrm{d^ny}}{\mathrm{dx^n}}+$

$$k_1\,(ax+b)^{n-1}\frac{\mathrm{d^{n-1}y}}{\mathrm{dx^{n-1}}}+\cdots+k_{n-1}\,(ax+b)\frac{\mathrm{dy}}{\mathrm{dx}}+k_n\,y\ = F(x)$$

is called Legendre's linear equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$(ax+b) = e^t \quad \Rightarrow t = \log(ax+b)$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{a}{ax+b}$

$\Rightarrow (ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$, where $D \equiv \frac{d}{dt}$

Similarly$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$

$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$ and so on.

Example 30 Solve the differential equation:

$$(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1 \quad ....①$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(3x+2) = e^t \quad \therefore t = \log(3x+2)$

$\Rightarrow (3x+2)\frac{dy}{dx} = 3Dy$, $(3x+2)^2\frac{d^2y}{dx^2} = 3^2D(D-1)y$

Also $3x^2+4x+1 = \frac{1}{3}(9x^2+12x+3)$

$$= \frac{1}{3}((3x)^2 + 2.3.2x + 4 - 4 + 3)$$

$$= \frac{1}{3}((3x+2)^2 - 1)$$

$$= \frac{1}{3}(e^{2t} - 1)$$

∴① May be rewritten as

$(9D(D-1) + 9D - 36)y = \frac{1}{3}(e^{2t}-1)$

$\Rightarrow 9(D^2-4)y = \frac{1}{3}(e^{2t}-1)$

Auxiliary equation is: $9(m^2-4) = 0$

$\Rightarrow m = \pm 2$

C.F. = $c_1e^{-2t} + c_2e^{2t}$

$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$

$$\text{P.I.} = \frac{1}{f(D)} F(x) = \frac{1}{9(D^2-4)} \frac{1}{3}(e^{2t} - 1)$$

$$= \frac{1}{27}\left(\frac{1}{(D^2-4)} e^{2t} - \frac{1}{(D^2-4)} e^{0t}\right)$$

$$= \frac{1}{27}\left(\frac{t}{2.2} e^{2t} - \frac{1}{(0-4)} e^{0t}\right)$$, Putting $D = 2$ in 1st term, it is a case of failure $\therefore \frac{1}{(D^2-4)} e^{2t} = t \frac{1}{f'(2)} e^{2x}$, also $D = 0$ in the 2nd term.

$$\therefore \text{P.I.} = \frac{1}{27}\left(\frac{t}{4} e^{2t} + \frac{1}{4}\right)$$

$$= \frac{1}{27}\left(\frac{\log(3x+2)}{4}(3x+2)^2 + \frac{1}{4}\right)$$

$$= \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2 + \frac{1}{108}[(3x+2)^2 \log(3x+2) + 1]$$

Example 31 Solve the differential equation:

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 2\sin(\log(x+1)), \quad x > -1 \text{ ....①}$$

Solution: This is a Legendre's linear equation with variable coefficients.

Putting $(x+1) = e^t$ $\therefore$ t = $\log(x+1)$

$$\Rightarrow (x+1)\frac{dy}{dx} = Dy, (x+1)^2 \frac{d^2y}{dx^2} = 1^2 D(D-1)y$$

∴① May be rewritten as

$$(D(D-1) + D + 1)y = 2\sin t$$

$$\Rightarrow (D^2 + 1)y = 2\sin t$$

Auxiliary equation is: $(m^2 + 1) = 0$

$$\Rightarrow m = \pm i$$

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

$$= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1))$$

$$= \frac{c_1}{(3x+2)^2} + c_2(3x+2)^2$$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+1} 2\sin t$

$=2t\frac{1}{2D}\sin t$, Putting $D^2 = -1$, case of failure

$$\therefore \frac{1}{(D^2+1)}\sin t = t\frac{1}{f'(D)}\sin t$$

$$= t\int \sin t\, dt = -\mathrm{t}\cos t$$

$\therefore$ P.I$= -\log(x+1)\cos(\log(x+1))$

Complete solution is: y = C.F. + P.I

$$y = c_1\cos(\log(x+1)) + c_2\sin(\log(x+1)) - \log(x+1)\cos(\log(x+1))$$

11.5 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2nd and higher order differential equations with constant coefficients as well as variable coefficients.

Working rule

Consider a 2nd order linear differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = F(x) \;\ldots\ldots ①$$

1. Find complimentary function given as: C.F. = $c_1y_1 + c_2y_2$,

   where y_1 and y_2 are two linearly independent solutions of ①

2. Calculate $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, W is called Wronskian of y_1 and y_2

3. Compute $u_1 = -\int \frac{y_2F(x)}{W}dx$, $u_2 = \int \frac{y_1F(x)}{W}dx$
4. Find P.I. = $u_1y_1 + u_2y_2$
5. Complete solution is given by: y = C.F. + P.I

Note: Method is commonly used to solve 2nd order differential but it can be extended to solve differential equations of higher orders.

Example 32 Solve the differential equation: $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2+1)y = \operatorname{cosec} x$

Auxiliary equation is: $(m^2+1) = 0$

$\Rightarrow \ m = \pm i$

C.F. = $c_1 \cos x + c_2 \sin x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos x$ and $y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \sin x \ \operatorname{cosec} x \ dx = -\int 1 dx = -x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \cos x \ \operatorname{cosec} x \ dx = \int \cot x \, dx = log|\sin x|$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= -x \cos x + \sin x \ log|\sin x|$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ log|\sin x|$

Example 33 Solve the differential equation: $(D^2 - 2D + 1)y = e^x$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - 2m + 1) = 0$

$\Rightarrow \ m = 1,1$

C.F. = $(c_1 + c_2 x)e^x = c_1 e^x + c_2 x \, e^x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^x$ and $y_2 = x \, e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x \, e^x \\ e^x & x \, e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{x e^x e^x}{e^{2x}} dx = -\int x dx = -\frac{x^2}{2}$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x}{e^{2x}} dx = \int 1 dx = x$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$$= -\frac{x^2}{2}e^x + x^2 e^x = \frac{x^2}{2}e^x$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = (c_1 + c_2\, x)e^x + \frac{x^2}{2}e^x$$

Example 34 Solve the differential equation: $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + 4y = x \sin 2x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2 + 4)y = x \sin 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\frac{1}{2}\int x \sin^2 2x \; dx = -\frac{1}{4}\int x(1 - \cos 4x)\, dx$$

$$= -\frac{1}{4}\left[\frac{x^2}{2} - \left[(x)\left(\frac{\sin 4x}{4}\right) - (1)\left(-\frac{\cos 4x}{16}\right)\right]\right]$$

$$= \left[-\frac{x^2}{8} + \frac{x\ \sin 4x}{16} + \frac{\cos 4x}{64}\right]$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int x \sin 2x \cos 2x \; dx = \frac{1}{4}\int x \sin 4x \; dx$$

$$= \frac{1}{4}\left[(x)\left(-\frac{\cos 4x}{4}\right) - (1)\left(-\frac{\sin 4x}{16}\right)\right]$$

$$= \left[-\frac{x\ \cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$$= \cos 2x\left[-\frac{x^2}{8} + \frac{x\ \sin 4x}{16} + \frac{\cos 4x}{64}\right] + \sin 2x\left[-\frac{x\ \cos 4x}{16} + \frac{\sin 4x}{64}\right]$$

$= \frac{x}{16}(\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64}(\cos 4x \cos 2x + \sin 4x \sin 2x)$

$- \frac{x^2}{8}\cos 2x \quad = \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Example 35 Solve the differential equation: $(D^2 - D - 2)y = e^{(e^x+3x)}$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - m - 2) = 0$

$\Rightarrow \; m = -1,2$

C.F. = $c_1 e^{-x} + c_2\, e^{2x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ and $y_2 = \; e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{e^{2x} e^{(e^x+3x)}}{3e^x}\, dx = \; -\int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$

$= -\frac{1}{3}\int e^{4x} e^{e^x} dx$, Putting $e^x = t \Rightarrow e^x dx = tdt$

$u_1 = -\frac{1}{3}\int t^3 e^t dt = -\frac{1}{3}[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$

$\Rightarrow u_1 = -\frac{e^{e^x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x+3x)}}{3e^x}\, dx = \; \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx \; \frac{1}{3}\int e^x e^{e^x} dx \; = \frac{e^{e^x}}{3}$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= -\frac{e^{e^x} e^{-x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$

$= \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Complete solution is: $y = $ C.F. + P.I

$$\Rightarrow y = c_1 e^{-x} + c_2\, e^{2x} + \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$$

Example 36 Given that $\therefore\ y_1 = x$ and $y_2 = \frac{1}{x}$ are two linearly independent solutions of the differential equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x, x \neq 0$

Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x}$

Given that $\therefore\ y_1 = x$ and $y_2 = \frac{1}{x}$

$\therefore$C.F. $= c_1y_1 + c_2y_2 = c_1x + \frac{c_2}{x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$$

$u_1 = -\int \frac{y_2F(x)}{W} dx = \int \frac{1}{x}.\frac{1}{x}.\frac{x}{2}\, dx = \frac{1}{2}\int \frac{1}{x} dx = \frac{1}{2}\log x$

$u_2 = \int \frac{y_1F(x)}{W} dx = -\int x.\frac{1}{x}.\frac{x}{2}\, dx = -\frac{x^2}{4}$

$\therefore$ P.I$= u_1y_1 + u_2y_2$

$= \frac{x}{2}\log x - \frac{x}{4}$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1x + \frac{c_2}{x} + \frac{x}{2}\log x - \frac{x}{4}$

Example 37 Solve the differential equation: $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2\log x$

using method of variation of parameters.

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Given differential equation may be rewritten as

$(\mathrm{D}(\mathrm{D}-1)-4\mathrm{D}+6)y = te^{2t}$

$\Rightarrow (D^2-5D+6)y = te^{2t}$

Auxiliary equation is: $(m-2)(m-3) = 0$

$\Rightarrow \; m = 2,3$

C.F. = $c_1e^{2t}+c_2e^{3t} = c_1y_1+c_2y_2$

$\therefore y_1 = e^{2t}$ and $y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$u_1 = -\int \frac{y_2F(t)}{W}dt = -\int \frac{e^{3t}te^{2t}}{e^{5t}}\,dt = -\int tdt = -\frac{t^2}{2}$

$u_2 = \int \frac{y_1F(t)}{W}dt = \int \frac{e^{2t}te^{2t}}{e^{5t}}\,dt = \int te^{-t}dt = [(t)(-e^{-t})-(1)(e^{-t})]$

$= -te^{-t}-e^{-t}$

$\therefore$ P.I= $u_1y_1+u_2y_2$

$= -\frac{t^2}{2}e^{2t}-(te^{-t}+e^{-t})e^{3t}$

$= -\frac{t^2}{2}e^{2t}-te^{2t}-e^{2t} = -e^{2t}\left(\frac{t^2}{2}+t+1\right)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = c_1e^{2t}+c_2e^{3t}-e^{2t}\left(\frac{t^2}{2}+t+1\right)$

or $y = c_1x^2+c_2\,x^3-x^2\left(\frac{(\log x)^2}{2}+\log x+1\right)$

$\Rightarrow y = c_3x^2+c_2\,x^3-\frac{x^2}{2}(\log x)^2-x^2\log x,\ c_3 = c_1-1$

11.6 Solving Simultaneous Linear Differential Equations

Linear differential equations having two or more dependent variables with single independent variable are called simultaneous differential equations and can be of two types:

Type 1: $f_1(D)x + f_2(D)y = F(t)$, $g_1(D)x + g(D)y = G(t), D \equiv \frac{d}{dt}$

Consider a system of ordinary differential equations in two dependent variables x and y and an independent variable t:

$$f_1(D)x + f_2(D)y = F(t)\ ,\ \ g_1(D)x + g(D)y = G(t), D \equiv \frac{d}{dt}$$

Given system can be solved as follows:

1. Eliminate y from the given system of equations resulting a differential equation exclusively in x.
2. Solve the differential equation in x by usual methods to obtain x as a function of t .
3. Substitute value of x and its derivatives in one of the simultaneous equations to get an equation in y.
4. Solve for y by usual methods to obtain its value as a function of t.

Example 38 Solve the system of equations: $\frac{dx}{dt} + y = e^t,\ \frac{dy}{dt} - x = e^{-t}$

Solution: Rewriting given system of differential equations as:

$Dx + y = e^t$……①
$Dy - x = e^{-t}$ …..②, $D \equiv \frac{d}{dt}$
Multiplying ① by D
$\Rightarrow D^2x + Dy = e^t$ …..③
Subtracting ② from ③, we get
$(D^2 + 1)x = e^t - e^{-t}$ …….④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m = \pm i$

C.F. = $c_1 \cos t + c_2 \sin t$

P.I. = $\frac{1}{f(D)} F(t) = \frac{1}{D^2+1}(e^t - e^{-t}) = \frac{1}{D^2+1}e^t - \frac{1}{D^2+1}e^{-t}$

$= \frac{1}{2}e^t - \frac{1}{2}e^{-t}$, Putting $D = 1$ and $D = -1$ in 1st and 2nd terms respectively

$\therefore x = c_1 \cos t + c_2 \sin t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$…….⑤

Using ⑤in ① $\Rightarrow D\left[c_1 \cos t + c_2 \sin t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}\right] + y = e^t$

$\Rightarrow \left[-c_1 \sin t + c_2 \cos t + \frac{1}{2}e^t + \frac{1}{2}e^{-t}\right] + y = e^t$

$\Rightarrow y = c_1 \sin t - c_2 \cos t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$ ……⑥

⑤ and ⑥ give the required solution.

Example 39 Solve the system of equations: $t\frac{\mathrm{d}x}{\mathrm{d}t} + y = 0, \ \frac{\mathrm{d}y}{\mathrm{d}t} + x = 0$

given that $x(1) = 1, \ y(-1) = 0$

Solution: Given system of equations is:

$t\frac{dx}{dt} + y = 0$……①

$t\frac{dy}{dt} + x = 0$ ……②,

Multiplying ① by $t\frac{d}{dt}$

$t\frac{d}{dt}(t\frac{dx}{dt} + y) = 0$

$\Rightarrow t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$ ……③

Subtracting ② from ③, we get

$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{dx}{dt} - x = 0$…….④

which is Cauchy's linear differential equation in x with variable coefficients.

.Putting $t = e^k \quad \therefore \log t = k$

$\Rightarrow t\frac{dx}{dt} = Dx, t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \mathrm{D}(\mathrm{D}-1)x \ , D \equiv \frac{d}{dk}$

$\therefore$ ④ may be rewritten as

$(\mathrm{D}(\mathrm{D}-1) + D - 1)x = 0$ ……⑤

$\Rightarrow (D^2 - 1)x = 0$

To solve ⑤ for x, Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. = $c_1e^k + c_2e^{-k} = c_1t + \frac{c_2}{t}$

$\therefore x = c_1 t + \frac{c_2}{t}$.......⑥

Using ⑥in ① $\Rightarrow t\frac{d}{dt}\left(c_1 t + \frac{c_2}{t}\right) + y = 0$

$\Rightarrow c_1 t - \frac{c_2}{t} + y = 0$

$\Rightarrow y = -c_1 t + \frac{c_2}{t}$ ....⑦

Also given that at $t = 1, x = 1$ and at $t = -1, y = 0$

Using in ⑥and ⑦ $c_1 + c_2 = 1,\ \ c_1 - c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2}$

Using $c_1 = c_2 = \frac{1}{2}$ in ⑥and ⑦, we get

$x = \frac{1}{2}\left(t + \frac{1}{t}\right),\ y = \frac{1}{2}\left(\frac{1}{t} - t\right)$

Example 40 Solve the system of equations:

$\frac{d^2x}{dt^2} + y = \sin t,\ \ \frac{d^2y}{dt^2} + x = \cos t$

Solution: Rewriting given system of differential equations as:

$D^2x + y = \sin t$......①
$D^2y + x = \cos t$ .....②, $D \equiv \frac{d}{dt}$
Multiplying ① by D^2
$D^2(D^2x + y) = D^2 \sin t$
$\Rightarrow D^4x + D^2y = -\sin t$ .....③
Subtracting ② from ③, we get
$(D^4 - 1)x = -\sin t - \cos t$ ....④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^4 - 1 = 0$

$\Rightarrow (m^2 - 1)(m^2 + 1) = 0$

$\Rightarrow m = \pm 1, \pm i$

C.F. = $c_1 e^t + c_2 e^{-t} + (c_3 \cos t + c_4 \sin t)$

P.I. = $\frac{1}{f(D)} F(t) = \frac{1}{D^4 - 1}(-\sin t - \cos t) = -\frac{1}{D^4 - 1}\sin t - \frac{1}{D^4 - 1}\cos t$

Putting $D^2 = -1$ i.e. $D^4 = 1$ in 1st and 2nd terms, it is a case of failure

$\therefore$ P.I. $= -t\frac{1}{4D^3}\sin t - t\frac{1}{4D^3}\cos t$

$= \frac{t}{4}\frac{1}{D}\sin t + \frac{t}{4}\frac{1}{D}\cos t$ putting $D^2 = -1$

$= -\frac{t}{4}\cos t + \frac{t}{4}\sin t$

$\therefore x = (c_1e^t + c_2e^{-t}) + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)$.......⑤

Using ⑤in ①

$\Rightarrow D^2\left[c_1e^t + c_2e^{-t} + (c_3\cos t + c_4\sin t) + \frac{t}{4}(\sin t - \cos t)\right] + y = \sin t$

$$\Rightarrow D\left[c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)\right] + y = \sin t$$

$$\Rightarrow \left[c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)\right] + y = \sin t$$

$\Rightarrow y = -(c_1e^t + c_2e^{-t}) + (c_3\cos t + c_4\sin t) + \left(\frac{t}{4} + \frac{1}{2}\right)(\sin t - \cos t)$ ...⑥

⑤ and ⑥ give the required solution.

Type II: Symmetric simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Simultaneous differential equations in the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ can be solved by the method of grouping or the method of multipliers or both to get two independent solutions: $u = c_1, v = c_2$; where c_1 and c_2 are arbitrary constants.

Method of grouping: In this method, we consider a pair of fractions at a time which can be solved for an independent solution.

Method of multipliers: In this method, we multiply each fraction by suitable multipliers (not necessarily constants) such that denominator becomes zero.

If a, b, c are multipliers, then $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{adx+bdy+cdz}{aP+bQ+cR}$

Example 41 Solve the set of simultaneous equations:

$$\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$$

Taking x, y, z as multipliers, each fraction equals

$$\frac{xdx+ydy+zdz}{(xz^2-2xyz-xy^2+xy^2+xyz+xyz-xz^2)} = \frac{xdx+ydy+zdz}{0}$$

$\Rightarrow \; xdx + ydy + zdz = 0$

Integrating, we get $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c'_1$

1st independent solution is: $u = x^2 + y^2 + z^2 = c_1$……①

Now for 2nd independent solution, taking last two members of the set of equations: $\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$

$\Rightarrow (y-z)dy = (y+z)dz$

$\Rightarrow \; ydy - (zdy + ydz) - zdz = 0$

$\Rightarrow \; ydy - d(yz) - zdz = 0$

Integrating, we get

$\frac{y^2}{2} - yz - \frac{z^2}{2} = c'_2$

$\Rightarrow \quad v = y^2 - 2yz - z^2 = c_2$ ……….②

① and ② give the required solution.

Exercise 11B

Q1. Solve the following differential equations:

i. $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Ans. $\langle y = (c_1 + c_2\log x)x^2 + \frac{1}{4} + 2x + \frac{x^2}{2}(\log x)^2\rangle$

ii. $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$

Ans. $\langle y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}\rangle$

iii. $(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$

Ans. $\langle y = c_1(2x+3)^{\frac{3+\sqrt{57}}{4}} + c_2(2x+3)^{\frac{3-\sqrt{57}}{4}} - \frac{3}{14}(2x+3) + \frac{3}{4}\rangle$

iv. $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + y = 4\cos(\log(x+1))$

Ans.$\langle y = c_1\cos(\log(x+1)) + c_2\, sin(\log(x+1)) + 2\log(x+$
$1.sin\log x+1$

Q2. Solve the following differential equations using method of variation of parameters

i. $\frac{d^2y}{dx^2} + y = xsinx$

Ans. $\langle y = c_1\cos x + c_2\sin x + \frac{1}{8}\cos x + \frac{x}{4}\sin x - \frac{x^2}{4}\cos x\rangle$

ii. $(D^2 - 1)y = e^{-2x}\sin e^{-x}$

Ans. $\langle y = c_1e^x + c_2e^{-x} - \sin e^{-x} - e^x\cos e^{-x}\rangle$

iii. $(D^2 - 2D)y = e^x\sin x$

Ans. $\langle y = c_1 + c_2e^{2x} - \frac{1}{2}e^x\sin x\rangle$

iv. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x\log x$

Ans: $\langle y = c_1 + c_2e^{2x} + \frac{x^2}{4}e^x(2\log x - 3)\rangle$

Q2. Solve the following set of simultaneous differential equations

i. $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$

Ans:$\langle x = e^{6t}(c_1\cos t + c_2\sin t), y = e^{6t}(c_1 - c_2)\cos t + (c_1 + c_2)\sin t)\rangle$

ii. $(D+1)x + (2D+1)y = e^t,\ (D-1)x + (D+1)y = 1$

Ans: $\langle x = c_1e^t + c_2e^{-2t} + 2e^{-t}, y = 3c_1e^t + 2c_2e^{-2t} + 3e^{-t}\rangle$

iii. $\frac{dx}{(z^2-2yz-y^2)} = \frac{dy}{(xy+zx)} = \frac{dz}{(xy-zx)}$

Ans: $\langle xy - z = c_1 ,\ x^2 - y^2 + z^2 = c_2\rangle$

11.7 Previous Years Solved Questions

Q1. Solve $(D^2 + D + 1)^2(D-2)y = 0$

$\langle \boldsymbol{Q1(h), GGSIPU,\ December\ 2012}\rangle$

Solution: Auxiliary equation is: $(m^2 + m + 1)^2(m-2)y = 0$……①

Solving ①, we get

$\Rightarrow\ m = 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

C.F. = $c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3\,x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3\,x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Q2. Solve $(D^2 - 1)y = \cosh x \cos x$

$\langle \boldsymbol{Q8(b),\ GGSIPU,\ December\ 2012}\rangle$

Solution: Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. = $c_1e^x + c_2e^{-x}$

P.I. = $\frac{1}{f(D)}\ F(x)$

$= \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\cos x\right)$ $\qquad\because \cosh x = \frac{e^x+e^{-x}}{2}$

$= \frac{1}{D^2-1}\left(\frac{e^x}{2}\cos x + \frac{e^{-x}}{2}\cos x\right)$

$= \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x + \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1}\cos x$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D)}\cos x + \frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{2D-1}\cos x + \frac{e^{-x}}{2}\frac{1}{-2D-1}\cos x \qquad \text{Putting } D^2 = -1$$

$$= \frac{e^x}{2}\frac{2D+1}{4D^2-1}\cos x - \frac{e^{-x}}{2}\frac{2D-1}{4D^2-1}\cos x$$

$$= -\frac{e^x}{10}(2D+1)\cos x + \frac{e^{-x}}{10}(2D-1)\cos x \quad \text{Putting } D^2 = -1$$

$$= -\frac{e^x}{10}(-2\sin x + \cos x) + \frac{e^{-x}}{10}(-2\sin x - \cos x)$$

$$\therefore \text{P.I.} = \frac{e^x}{10}(2\sin x - \cos x) - \frac{e^{-x}}{10}(2\sin x + \cos x)$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1e^x + c_2e^{-x} + \frac{e^x}{10}(2\sin x - \cos x) - \frac{e^{-x}}{10}(2\sin x + \cos x)$$

Q3. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by the method of variation of parameters.

$\langle \boldsymbol{Q9(a), \; GGSIPU, \; December\; 2012} \rangle$

Solution: $\Rightarrow (D^2+4)y = 4\tan 2x$

Auxiliary equation is: $(m^2+4) = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1\cos 2x + c_2\sin 2x = c_1y_1 + c_2y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2F(x)}{W}dx = -\frac{4}{2}\int \sin 2x \tan 2x \; dx = -2\int \frac{\sin^2 2x}{\cos 2x}\; dx$$

$$-2\int \frac{1-\cos^2 2x}{\cos 2x}\; dx = -2\int(\sec 2x - \cos 2x)\, dx$$

$$= -2\left[\frac{1}{2}\log|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right]$$

$= [\sin 2x - \log|\sec 2x + \tan 2x|]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int 4 \tan 2x \cos 2x \; dx = \; 2\int \sin 2x \; dx$

$= -\cos 2x$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= \cos 2x\, [\sin 2x - \log|\sec 2x + \tan 2x|] - \sin 2x \cos 2x$

$= -\cos 2x \log|\sec 2x + \tan 2x|$

Complete solution is: $y \; = \;$ C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log|\sec 2x + \tan 2x|$

Q4. Solve the system of equations: $\frac{dx}{dt} + x \; = y + e^t, \;\; \frac{dy}{dt} + y \; = x + e^t$

$\langle \boldsymbol{Q9(b), \; GGSIPU, \;\; December \; 2012} \rangle$

Solution: Rewriting given system of differential equations as:

$(D+1)x - y = e^t$……①
$(D+1)y - x = e^t$ …..②, $D \equiv \frac{d}{dt}$
Multiplying ① by $(D+1)$
$\Rightarrow (D+1)^2 x - (D+1)y = (D+1)e^t$
$(D^2 + 2D + 1)x - (D+1)y = 2e^t$…..③
Adding ② and ③, we get
$(D^2 + 2D)x = 3e^t$ …….④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 2m = 0$

$\Rightarrow m = 0, -2$

C.F. $= \; c_1 + c_2 e^{-2t}$

P.I. $= \frac{1}{f(D)} \; F(t) \; = 3\frac{1}{D^2+2D} e^t$

$= e^t$ Putting $D = 1$

$\therefore x = c_1 + c_2 e^{-2t} + e^t$…….⑤

Using ⑤in ① $\Rightarrow D[c_1 + c_2e^{-2t} + e^t] + c_1 + c_2e^{-2t} + e^t - y = e^t$

$\Rightarrow -2c_2e^{-2t} + e^t + c_1 + c_2e^{-2t} - y = 0$

$\Rightarrow y = c_1 - c_2e^{-2t} + e^t$ ……⑥

⑤ and ⑥ give the required solution.

Q5. Solve by method of variation of parameters $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2013} \rangle, \langle \boldsymbol{Q3(b), GGSIPU, 2^{nd} term\ 2014} \rangle$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$(m - 3)^2 = 0$

$\Rightarrow m = 3,3$

C.F. = $(c_1 + c_2\, x)e^{3x} = c_1e^{3x} + c_2x\, e^{3x} = c_1y_1 + c_2y_2$

$\therefore y_1 = e^{3x}$ and $y_2 = xe^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

$u_1 = -\int \frac{y_2F(x)}{W} dx = -\int \frac{xe^{3x}e^{3x}}{x^2e^{6x}} dx = -\int \frac{1}{x} dx = -\log x$

$u_2 = \int \frac{y_1F(x)}{W} dx = \int \frac{e^{3x}e^{3x}}{x^2e^{6x}} dx = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$\therefore$ P.I= $u_1y_1 + u_2y_2$

$= -e^{3x}\log x - e^{3x} = -e^{3x}(1 + \log x)$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2x)e^{3x} - e^{3x}(1 + \log x)$

Q6. Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2013} \rangle$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m + 1)^2 = 0$

$\Rightarrow \ m = 0, -1, -1$

C.F. = $c_1 + \ e^{-x}(c_2 + c_3x)$

P.I. = $\frac{1}{f(D)} \ F(x) \ = \frac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$

$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{-4D - 8 + D} \sin 2x$, putting $D = 2$ in 1st term, $D^2 = -4$ in the 2nd term

$= \frac{1}{18} e^{2x} - \frac{3D - 8}{(3D + 8)(3D - 8)} \sin 2x = \frac{1}{18} e^{2x} - \frac{3D - 8}{(9D^2 - 64)} \sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{100}(3\text{D} - 8)\sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Complete solution is: $y \ = \$ C.F. + P.I

$\Rightarrow y \ = c_1 + \ e^{-x}(c_2 + c_3x) + \ \frac{1}{18} e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Q7. Solve $(D^2 - 2D + 1)y = xe^x \cos x$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2014} \rangle$

Solution: Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m - 1)^2$

$\Rightarrow m = 1, 1$

C.F. = $(c_1 + c_2x)e^x$

P.I. $= \frac{1}{f(D)} \ F(x) \ = \frac{1}{D^2 - 2D + 1} xe^x \cos x$

$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \cos x$

$= e^x \frac{1}{D^2} x \cos x$

$= e^x \frac{1}{D} \int x \cos x \, dx$

$= e^x \frac{1}{D} [(x)(\sin x) - (1)(-\cos x)]$

$= e^x \frac{1}{D} [x \sin x + \cos x]$

$= e^x [\int x \sin x \, dx + \int \cos x \, dx]$

$= e^x \big[[(x)(-\cos x) - (1)(-\sin x)] + \sin x\big]$

$\therefore$ P.I. $= e^x [-x \cos x + 2 \sin x]$

Complete solution is: $y \ = \ $ C.F. + P.I

$\Rightarrow y \ = (c_1 + c_2 x) e^x + e^x [-x \cos x + 2 \sin x]$

Q8. Solve by M.O.V.P. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y \ = e^x \log x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2014} \rangle$

Solution: Given differential equation may be rewritten as

$(D^2 - 2D + 1)y = e^x \log x$

: Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2$

$\Rightarrow m = 1,1$

C.F. = $(c_1 + c_2 x) e^x = c_1 y_1 + c_2 y_2$
$\therefore y_1 = e^x$ and $y_2 = x e^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x\, e^x \\ e^x & x\, e^x + \ e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{x\, e^x\, e^x \log x}{e^{2x}} \ dx = \ -\int x \log x \, dx$

$\int x \log x \, dx = I \ = \left[(x)(x \log x - x) - (1)\left(I - \frac{x^2}{2}\right)\right]$

$\because \int \log x \, dx = x \log x - x$

$\Rightarrow 2I = x^2 \log x - x^2 + \frac{x^2}{2}$

$$\Rightarrow I = \int x \log x \, dx \;=\; \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$$\therefore u_1 = \frac{x^2}{4} - \frac{x^2}{2}\log x$$

$$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x \log x}{e^{2x}} \, dx = \int \log x \, dx = \; x \log x - x$$

$$\therefore \text{P.I} = \left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right) e^x + (x \log x - x) x e^x$$

$$= e^x \left(\frac{x^2}{4} - \frac{x^2}{2}\log x + x^2 \log x - x^2\right)$$

$$= \frac{x^2}{2} e^x \left(\log x - \frac{3}{2}\right)$$

Complete solution is: $y \;=\;$ C.F. + P.I

$$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{x^2}{2} e^x \left(\log x - \frac{3}{2}\right)$$

Q9. Solve $(D-1)^2(D+1)^2 = \sin^2 \frac{x}{2} + e^x + x$

$\langle \boldsymbol{Q1(a), GGSIPU, December\ 2015} \rangle$

Solution: Auxiliary equation is: $(m-1)^2(m+1)^2 = 0$

$\Rightarrow \; m = 1,1,-1,-1$

C.F. = $(c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x}$

P.I. = $\frac{1}{f(D)} \; F(x) \; = \frac{1}{((D-1)(D+1))^2}(\sin^2 \frac{x}{2} + e^x + x)$

$$= \frac{1}{2}\frac{1}{D^4 - 2D^2 + 1}(1 - \cos x) + \frac{1}{D^4 - 2D^2 + 1} \; e^x + \frac{1}{D^4 - 2D^2 + 1} x$$

$$= \frac{1}{2}\frac{1}{D^4 - 2D^2 + 1} \; e^{0x} - \frac{1}{2}\frac{1}{D^4 - 2D^2 + 1}\cos x + \frac{1}{D^4 - 2D^2 + 1} \; e^x + \frac{1}{D^4 - 2D^2 + 1} x$$

Now $\frac{1}{2}\frac{1}{D^4 - 2D^2 + 1} \; e^{0x} = \frac{1}{2}$, putting $D = 0$

Also $\frac{1}{2}\frac{1}{D^4 - 2D^2 + 1}\cos x = \frac{1}{8}\cos x$ putting $D^2 = -1$

Again $\frac{1}{D^4 - 2D^2 + 1} \; e^x \; = \; x \frac{1}{4D^3 - 4D} \; e^x$ as $f(1) = 0$, a case of failure 2 times

$= x^2 \frac{1}{12D^2-4} e^x = \frac{x^2}{8} e^x$, putting $D = 1$

And $\frac{1}{D^4-2D^2+1} x = \frac{1}{1+(D^4-2D^2)} x = [1 + (D^4 - 2D^2)]^{-1} x = x$

$\therefore$ P.I.$= \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8} e^x + x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} - \frac{1}{8}\cos x + \frac{x^2}{8} e^x + x + \frac{1}{2}$

Q.10 Solve $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^4 \sin x$

$\langle \boldsymbol{Q3(b), GGSIPU, December\ 2015} \rangle$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2 \frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Equation may be rewritten as

$(D(D-1) - 4D + 6)y = e^{4t} \sin e^t$

$\Rightarrow (D^2 - 5D + 6)y = e^{4t} \sin e^t$, $D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow m = 2,3$

C.F. $= c_1 e^{2t} + c_2 e^{3t}$

$= c_1 x^2 + c_2 x^3$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2-5D+6} e^{4t} \sin e^t$

$= e^{4t} \frac{1}{(D+4)^2-5(D+4)+6} \sin e^t$

$= e^{4t} \frac{1}{D^2+3D+2} \sin e^t = e^{4t} \frac{1}{(D+1)(D+2)} \sin e^t$

$=e^{4t}\left[\frac{1}{(D+1)}-\frac{1}{(D+2)}\right]\sin e^{t}= e^{4t}\left[\frac{1}{(D+1)}\sin e^{t}-\frac{1}{(D+2)}\sin e^{t}\right]$

$=e^{4t}[e^{-t}\int e^{t}\sin e^{t}\,dt-e^{-2t}\int e^{2t}\sin e^{t}\,dt]$

$\because \frac{1}{(D+a)}F(t)=e^{-at}\int e^{at}F(t)dt$

$= e^{4t}[e^{-t}(-cose^{t})-e^{-2t}(-e^{t}\cos e^{t}+\sin e^{t}]$

Solving the two integrals by putting $e^{t}=u, \therefore e^{t}dt=du$

$\therefore$ P.I $=-e^{2t}\sin e^{t}=-x^{2}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1x^2+c_2x^3-x^2\sin x$